TASK MONOGRAPH SERIES ON SCIENCE AND TECHNOLOGY

Khrystyna P. Gnatenko, Volodymyr M. Tkachuk

The soccer-ball problem in quantum space

A Monograph

$$[X,P] = i\hbar(1+\beta P^2)$$

$$\sqrt{\beta}m = \text{const}$$

$$[X_i, X_j] = i\hbar\theta_{ij}$$

$$\theta m = \text{const}$$

$$[X_i, X_j] = i\hbar\theta^k_{ij}X_k$$

$$[X_i, X_j] = i\hbar\varepsilon_{ijk}p^a_k$$

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1. Introduction

Quantum spaces described by deformed commutation relations for coordinates and momenta are considered in the monograph. Studies of different physical systems in the frame of deformed algebras give a possibility to find the effects of space quantization on their properties and to estimate the minimal length. The monograph is devoted to studies of the problem of a macroscopic body which is known as the soccer-ball problem in quantum space. A solution of this problem is important for the self-consistency of the quantum space theory and also for finding new effects of space quantization in a wide class of physical systems including composite systems, macroscopic bodies. We show that in the frame of different algebras the relation of parameters of the algebras with mass opens a possibility to solve a list of problems including the problem of motion of a macroscopic body, the problem of violation of the weak equivalence principle, the problem of violation of the properties of kinetic energy, the problem of the dependence of the Galilean and Lorentz transformations on mass.

The monograph is organized as follows. Deformed algebras leading to the minimal length are presented in Chapter 2. Three types of deformed algebras (algebras with nonlinear deformation, algebras of a canonical type, algebras of the Lie type) are considered in details. The relation of nonlinear deformed algebras with linear ones is presented. The problem of violation of the rotational and time reversal symmetries in the frame of a noncommutative algebra of the canonical type is also discussed. We construct a noncommutative algebra which is rotationally invariant, time reversal invariant, and in addition, equivalent to a noncommutative algebra of a canonical type.

In Chapter 3 the problem of a macroscopic body is examined in the frame of a nonlinear deformed algebra leading to a minimal length. We show that if we assume that the parameters of a deformed algebra are the same for elementary particles and macroscopic bodies the great effect of minimal length on the motion of macroscopic bodies is obtained. We find that the motion of a macroscopic body in a deformed space is described by the effective parameter of deformation which is less than the parameters of deformations corresponding to elementary particles. We conclude that if the parameter of deformation is related to mass, the problem of the macroscopic body is solved in the deformed space with the minimal length, the properties of the kinetic energy are recovered, the weak equivalence principle is preserved, the Galilean and Lorentz transformations are the same for particles (bodies) with different masses.

The features of a description of motion of the center-of-mass of a composite system (a macroscopic body) in a noncommutative phase space of a canonical type are presented in Chapter 4. We show that the motion of the center-of-mass of a composite system is described by the effective parameters of noncommutativity and this motion is not independent of the relative motion. We conclude that if we consider the parameter of coordinate noncommutativity to be proportional to mass and the parameter of momentum noncommutativity to be inversely proportional to mass, the two-particle problem can be reduced to a one-particle problem, the kinetic energy is additive and does not depend on the composition, the weak equivalence principle is preserved in the noncommutative phase space of the canonical type.

In Chapter 5 the results presented in Chapter 4 are generalized to the case of a rotationally and time reversal invariant noncommutative algebra of a canonical type. We show that if the tensors of noncommutativity depend on mass in a special way, the commutation relations for coordinates and momenta of the center-of-mass reproduce the relations of a noncommutative algebra with effective tensors of noncommutativity, and the weak equivalence principle is preserved in the rotationally and time reversal invariant noncommutative phase space.

In Chapter 6 we show that the relation of parameters of a noncommutative algebra with mass is also important in spaces with the Lie algebraic noncommutativity. Due to this relation the noncommutative algebra for the coordinates and the momenta of the center-of-mass is an algebra of the Lie type and the weak equivalence principle is preserved in the frame of different noncommutative algebras of the Lie type.

The conclusions are presented in Chapter 7.

2. Deformations of commutation relations for coordinates and momenta leading to space quantization

The idea that coordinates may satisfy deformed commutation relations was proposed by Heisenberg for solving the problem of ultraviolet divergences in the quantum field theory. This idea was formalized by Snyder in his paper in 1947 [1].

The recently growing interest in studies of deformed algebras of different types is motivated by the development of the string theory and the quantum gravity (see, for example, [2–8]) which predicts the existence of a nonzero minimum uncertainty in position (minimal length).

Many different algebras have been considered to describe a quantum space (space with minimal length). These algebras can be divided into three types: noncommutative algebras of a canonical type (commutators for coordinates and momenta are equal to constants), noncommutative algebras of the Lie type (commutators for coordinates and momenta are equal to linear functions of coordinates and momenta), nonlinear deformed algebras (commutators for coordinates and momenta are equal to nonlinear functions of coordinates and momenta).

In this chapter we present well studied deformed algebras describing the quantum space. Section 2.1 is devoted to nonlinear deformed algebras leading to minimal length. Noncommutative algebras of a canonical type are presented in Section 2.2. Problems of the rotational symmetry breaking and the time reversal symmetry breaking are discussed in this section. In Section 2.3 a noncommutative algebra which is rotationally and time reversal invariant and moreover equivalent to a noncommutative algebra of a canonical type is presented. In Section 2.4 different cases of noncommutative algebras of the Lie type are considered. In Section 2.5 it is shown that nonlinear deformed algebras are related to linear ones.

2.1. Nonlinear deformed algebras

An important prediction of investigations in the string theory and the quantum gravity is the existence of a nonzero minimum uncertainty in the position, minimal length, which follows from the generalized uncertainty principle

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P}+\beta\Delta P\right) \tag{1}$$

where β is a constant. The minimum uncertainty in position

$$\Delta X_{min}=\hbar\sqrt{\beta} \tag{2}$$

follows from the generalized uncertainty principle (1) and it is considered to be of the order of the Planck length $l_P=\sqrt{\hbar G/c^3}=1.6\times10^{-35}$m (see, for instance, [4]).

The relation (1) can be obtained in the one-dimensional case considering a commutator for the coordinate and the momentum to be deformed as [9–11]

$$[X,P]=i\hbar(1+\beta P^2) \tag{3}$$

The parameter β is called a parameter of deformation ($\beta \geq 0$). For $\beta \to 0$, the relation (3) reduces to an ordinary commutation relation. Using the Heisenberg uncertainty principle from the commutation relation (3) we obtain (1) with the notations $\sqrt{\langle\Delta X^2\rangle}\to\Delta X$, $\sqrt{\langle\Delta P^2\rangle}\to\Delta P$.

The coordinates and the momenta which satisfy (3) can be represented as

$$X=x \tag{4}$$

$$P=\frac{1}{\sqrt{\beta}}\tan(\sqrt{\beta}p) \tag{5}$$

where the operators x, p satisfy the relation $[x,p]=i\hbar$.

A generalization of the algebra (3) to

$$[X,P]=i\hbar(1+\alpha X^2+\beta P^2) \tag{6}$$

($\alpha \geq 0,\ \beta \geq 0$, and $\alpha\beta < \hbar^{-2}$) leads to minimum uncertainties in the position and momentum $\Delta X_0 = \hbar\sqrt{\beta/(1-\hbar^2\alpha\beta)}$ and $\Delta P_0 = \hbar\sqrt{\alpha/(1-\hbar^2\alpha\beta)}$ [12, 11]. It should be mentioned that in the frame of this algebra the spectrum of the harmonic oscillator can be found exactly [13–15]. Note that if $\beta=0$, the algebra (6) and also a more general one

$$[X,P]=i\hbar g(X) \tag{7}$$

(where $g(X)$ is a deformation function) describe a particle with the position dependent effective mass [14, 16, 17].

In a more general case of space with minimal length, the commutation relation for the coordinate and the momentum can be written introducing the function of deformation $f(P)$, namely

$$[X,P]=i\hbar f(P) \tag{8}$$

where the function of deformation is strictly positive ($f>0$), the domain of P in the momentum representation is $-a\leq P\leq a$. For an invariance of (8) with respect to the reflection ($X\to -X$, $P\to -P$) and for the time reversal invariance[1] function $f(P)$ has to be even, $f(-P)=f(P)$. To recover the usual commutation relations, for $\beta=0$ the function reads $f(0)=1$. It is worth mentioning that different deformation functions have been considered to describe the quantum space (see, for instance, [19–25]).

The relation (8) leads to the minimal length [26]

$$\Delta X_{min}=\frac{\pi\hbar}{2}\left(\int\limits_0^a\frac{dP}{f(P)}\right)^{-1} \tag{9}$$

Note that the minimal length exists if the integral $\int_0^a dP/f(P)$ is finite, otherwise, the minimal length is equal to zero. The equality (9) allows calculating the minimal length for the arbitrary function of deformation. Applying this formula to the deformed function in (3) we recover the result for the minimal length (2).

The one-dimensional deformed algebra (3) can be generalized to cases with higher dimensions such as

$$[X_i,X_j]=i\hbar\frac{(2\beta-\beta')+(2\beta+\beta')\beta P^2}{1+\beta P^2}(P_iX_j-P_jX_i) \tag{10}$$

$$[X_i,P_j]=i\hbar(\delta_{ij}(1+\beta P^2)+\beta' P_iP_j) \tag{11}$$

$$[P_i,P_j]=0 \tag{12}$$

where $\beta\geq 0$, $\beta'\geq 0$ are parameters of deformation (see, for instance, [27–36]). The space with the algebra (10)–(12) is characterized by the minimal length $\hbar\sqrt{\beta+\beta'}$.

---

1. Upon time reversal $X\to X$, $P\to -P$. In quantum mechanics a time-reversal operation involves a complex conjugation[18]

The coordinates and the momenta which satisfy (10)–(12) can be represented as [29]

$$X_i = (1+\beta p^2)x_i + \beta' p_i p_j x_j \tag{13}$$

$$P_i = p_i \tag{14}$$

with the coordinates and the momenta x_i, p_i satisfying the relations of the undeformed algebra

$$[x_i, x_j] = [p_i, p_j] = 0 \tag{15}$$

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{16}$$

The relations (10)–(12) can be written in a more general form

$$[X_i, X_j] = G(P^2)(X_i P_j - X_j P_i) \tag{17}$$

$$[X_i, P_j] = f(P^2)\delta_{ij} + F(P^2)P_i P_j \tag{18}$$

$$[P_i, P_j] = 0 \tag{19}$$

For the consistency of the algebra (17)–(19) the Jacobi identity has to be satisfied for all possible triplets of operators. Therefore, the functions $G(P^2)$, $F(P^2)$, $f(P^2)$ have to satisfy the following relation [37]

$$\begin{aligned} &f(F-G) - 2f'(f+FP^2) = 0 \\ &f' = \frac{\partial f}{\partial P^2} \end{aligned} \tag{20}$$

Choosing $f = 1+\beta P^2$, $F = \beta'$, the relations (17)–(19) reduce to (10)–(12). If $f = 1$, $F = \beta$ the algebra (17)–(19) corresponds to the non-relativistic Snyder algebra (see, for instance, [1, 38–42])

$$[X_i, X_j] = i\hbar\beta(P_j X_i - P_i X_j) \tag{21}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta P_i P_j) \tag{22}$$

$$[P_i, P_j] = 0 \tag{23}$$

Note that due to the relation (17) the algebra (17)–(19) is not invariant with respect to translations in the configurational space. In a particular case of $G = 0$ the commutation relations (17)–(19) are transformed to

$$[X_i, X_j] = [P_i, P_j] = 0 \tag{24}$$

$$[X_i, P_j] = f(P^2)\delta_{ij} + F(P^2)P_i P_j \tag{25}$$

with the functions f and F satisfying the following relation

$$Ff - 2f'(f+FP^2) = 0 \tag{26}$$

The algebra (24)–(25) is invariant under translations in the configurational space.

In a particular case $f=\sqrt{1+\beta P^2}$ on the basis of the relation (26) we find $F=\beta\sqrt{1+\beta P^2}$ and then from (24), (25) we obtain the following algebra [43]

$$[X_i,X_j]=[P_i,P_j]=0 \tag{27}$$

$$[X_i,P_j]=i\hbar\sqrt{1+\beta P^2}(\delta_{ij}+\beta P_iP_j) \tag{28}$$

The representation for coordinates and momenta satisfying (27), (28) reads

$$X_i=x_i \tag{29}$$

$$P_i=\frac{p_i}{\sqrt{1-\beta p^2}} \tag{30}$$

Note also that the algebra which is invariant with respect to the translation can be obtained in the first order in β setting $\beta'=2\beta$ in (10)–(12). This algebra reads [44, 45]

$$[X_i,X_j]=[P_i,P_j]=0 \tag{31}$$

$$[X_i,P_j]=i\hbar(\delta_{ij}(1+\beta P^2)+2\beta P_iP_j) \tag{32}$$

2.2. Noncommutative algebras of canonical type

A two-dimensional noncommutative algebra of a canonical type is characterized by the following relations

$$[X_1,X_2]=i\hbar\theta \tag{33}$$

$$[X_1,P_1]=[X_2,P_2]=i\hbar \tag{34}$$

$$[P_1,P_2]=[X_1,P_2]=[X_2,P_1]=0 \tag{35}$$

where θ is a constant called the parameter of coordinate noncommutativity (see, for instance, [46–50]). The coordinates and momenta which satisfy the relations (33)–(35) can be represented by the coordinates and momenta x_i, p_i satisfying the usual commutation relations (15), (16). The representation reads

$$X_1=x_1-q\theta p_2 \tag{36}$$

$$X_2=x_2+(1-q)\theta p_1 \tag{37}$$

$$P_1=p_1,\quad P_2=p_2 \tag{38}$$

where q is a constant which can be arbitrary chosen. Traditionally, the symmetrical representation is considered, which corresponds to the case of $q=1/2$ and is as follows

$$X_1=x_1-\frac{\theta}{2}p_2 \tag{39}$$

$$X_2=x_2+\frac{\theta}{2}p_1 \tag{40}$$

$$P_1=p_1,\quad P_2=p_2 \tag{41}$$

From (33) the uncertainty relation follows

$$\Delta X_1 \Delta X_2 \geq \frac{\hbar|\theta|}{2} \tag{42}$$

where $\Delta X_i = \sqrt{\langle \Delta X_i^2 \rangle}$.

The eigenvalues of the squared length operator in the space (33)–(35)

$$R_{12}^2 = X_1^2 + X_2^2 \tag{43}$$

are

$$r_{n_{12}}^2 = 2\hbar|\theta| \left(n_{12} + \frac{1}{2} \right) \tag{44}$$

where n_{12} is a quantum number $n_{12} = 0, 1, 2, 3, \ldots$.[2] Therefore, for

$$\langle \Delta R_{12}^2 \rangle = \langle \Delta X_1^2 \rangle + \langle \Delta X_2^2 \rangle \tag{45}$$

$$\Delta R_{12} = \sqrt{\langle \Delta R_{12}^2 \rangle} \tag{46}$$

the following inequalities are satisfied

$$\begin{aligned} \langle \Delta R_{12}^2 \rangle &\geq \hbar|\theta| \\ \Delta R_{12} &\geq \sqrt{\hbar|\theta|} \end{aligned} \tag{47}$$

with $\langle X_1 \rangle = \langle X_2 \rangle = 0$. Thus, the algebra (33)–(35) leads to the minimal area $\hbar|\theta|$ and to the minimal length $\sqrt{\hbar|\theta|}$ [52].

It is worth noting that the noncommutativity of coordinates appears in the problem of a particle in a strong magnetic field. A particle with the charge e in a strong magnetic field $\mathbf{B}$ pointing in the X_3 direction moves on a noncommutative plane. The coordinates of the particle satisfy the following relation

$$[X_1, X_2] = -i\hbar \frac{c}{eB} \tag{48}$$

where c is the velocity of light [53–57].

In a more general case a noncommutative algebra of a canonical type is characterized by the following relations

$$[X_i, X_j] = i\hbar\theta_{ij} \tag{49}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \sigma_{ij}) \tag{50}$$

$$[P_i, P_j] = i\hbar\eta_{ij} \tag{51}$$

where θ_{ij}, η_{ij} are elements of constant antisymmetric matrixes called parameters of coordinate noncommutativity and parameters of momentum noncommutativity, σ_{ij} are constants, $i, j = (1, 2, 3)$ (see, for instance, [58–68]). Note that the parameters θ_{ij}, η_{ij}, σ_{ij} are constrained because of the Jacoby identity.

2. The details of calculations needed to obtain (44) are presented in [51, 52]

Considering a symmetrical representation for coordinates and momenta satisfying (49), (51)

$$X_i = x_i - \frac{1}{2}\sum_j \theta_{ij} p_j \tag{52}$$

$$P_i = p_i + \frac{1}{2}\sum_j \eta_{ij} p_j \tag{53}$$

with coordinates x_i and p_i satisfying the ordinary commutation relations (15), (16) and calculating the commutator $[X_i, P_j]$, we find

$$\sigma_{ij} = \sum_k \frac{\theta_{ik}\eta_{jk}}{4} \tag{54}$$

(see, for instance, [66, 69]).

The following uncertainty relations follow from the relations of the noncommutative algebra (49)–(51)

$$\langle \Delta X_i^2 \rangle \langle \Delta X_j^2 \rangle \geq \frac{\hbar^2 \theta_{ij}^2}{4} \tag{55}$$

$$\langle \Delta P_i^2 \rangle \langle \Delta P_j^2 \rangle \geq \frac{\hbar^2 \eta_{ij}^2}{4} \tag{56}$$

$$\langle \Delta X_i^2 \rangle \langle \Delta P_j^2 \rangle \geq \frac{\hbar^2 (\delta_{ij} + 2\sigma_{ij}\delta_{ij} + \sigma_{ij}^2)}{4} \tag{57}$$

Taking into account the inequality (55), we can write

$$\begin{aligned} \langle \Delta \mathbf{R}^2 \rangle^2 = \langle \Delta X_1^2 \rangle + \langle \Delta X_2^2 \rangle + \langle \Delta X_3^2 \rangle \geq \\ 2\langle \Delta X_1^2 \rangle \langle \Delta X_2^2 \rangle + 2\langle \Delta X_2^2 \rangle \langle \Delta X_3^2 \rangle + \\ 2\langle \Delta X_3^2 \rangle \langle \Delta X_1^2 \rangle \geq \frac{\hbar^2}{2}\left(\theta_{12}^2 + \theta_{23}^2 + \theta_{31}^2\right) \end{aligned} \tag{58}$$

Thus, the uncertainty relations (55) result in the existence of a restriction on the length in the noncommutative phase space

$$\Delta R \geq \left(\frac{\hbar^2}{2}\left(\theta_{12}^2 + \theta_{23}^2 + \theta_{31}^2\right)\right)^{\frac{1}{4}} \tag{59}$$

where $\Delta R = \sqrt{\langle \Delta \mathbf{R}^2 \rangle}$. Similarly from (56) for the length in the momentum space we have

$$\begin{aligned} &\langle \Delta \mathbf{P}^2 \rangle^2 = \langle \Delta P_1^2 \rangle + \langle \Delta P_2^2 \rangle + \langle \Delta P_3^2 \rangle \geq \\ &2\langle \Delta P_1^2 \rangle \langle \Delta P_2^2 \rangle + 2\langle \Delta P_2^2 \rangle \langle \Delta P_3^2 \rangle + 2\langle \Delta P_3^2 \rangle \langle \Delta P_1^2 \rangle \geq \\ &\frac{\hbar^2}{2}\left(\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2\right) \end{aligned} \tag{60}$$

$$\Delta P \geq \left(\frac{\hbar^2}{2} \left(\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2 \right) \right)^{\frac{1}{4}} \tag{61}$$

with $\Delta P = \sqrt{\langle \Delta \mathbf{P}^2 \rangle}$.

The restriction on the length in the noncommutative phase space can be also found considering the eigenvalues of the operator of the squared length. Using the representation (52) this operator reads

$$\begin{aligned} \mathbf{R}^2 = \sum_i X_i^2 = \mathbf{x}^2 + \frac{1}{4}[\boldsymbol{\theta} \times \mathbf{p}]^2 - (\boldsymbol{\theta} \cdot \mathbf{L}) = \\ \mathbf{x}^2 + \frac{1}{4}\theta^2 p^2 - \frac{1}{4}(\boldsymbol{\theta} \cdot \mathbf{p})^2 - (\boldsymbol{\theta} \cdot \mathbf{L}) \end{aligned} \tag{62}$$

where $\mathbf{x}^2 = \sum_i x_i^2$, and the components of the vector $\boldsymbol{\theta}$ are

$$\theta_k = \frac{1}{2} \sum_{i,j} \varepsilon_{ijk} \theta_{ij}$$

The first two terms in (62) are invariant under rotation. Therefore, for convenience, we choose a frame of reference with a coincidence of the x_3-axis direction with the direction of the vector $\boldsymbol{\theta}$ and write

$$\mathbf{R}^2 = \mathbf{x}^2 + \frac{1}{4}[\boldsymbol{\theta} \times \mathbf{p}]^2 - \theta(x_1 p_2 - x_2 p_1) = \tag{63}$$

$$x_1^2 + x_2^2 + x_3^2 + \frac{1}{4}\theta^2 p_1^2 + \frac{1}{4}\theta^2 p_2^2 - \theta(x_1 p_2 - x_2 p_1) \tag{64}$$

$$\theta = |\boldsymbol{\theta}| = \sqrt{\theta_{12}^2 + \theta_{23}^2 + \theta_{31}^2} \tag{65}$$

(the same notations for coordinates x_i in the chosen frame of reference are used in (64)). Note that $[x_3^2, \mathbf{R}^2] = 0$. Therefore, the eigenvalues of the squared length operator $\mathbf{R}^2$ are the following

$$R^2 = 2\hbar\theta \left(n + \frac{1}{2} \right) + r_3^2 \tag{66}$$

where n is a quantum number $n = 0, 1, 2...$, and r_3^2 denotes the eigenvalues of the operator x_3^2 [51, 52]. It follows from (66) that

$$\langle \Delta \mathbf{R}^2 \rangle \geq \hbar\theta \tag{67}$$

Therefore, the restriction on the value of length is given by the following inequality

$$\Delta R \geq \sqrt{\hbar\theta} \tag{68}$$

Note that the lower bound (68) is stronger than the bound given by (59).

Similarly on the basis of studies of eigenvalues of the squared length operator in the momentum space $\mathbf{P}^2 = \sum_i P_i^2$ is obtained [52]

$$\begin{aligned} &\Delta P \geq \sqrt{\hbar\eta} \\ &\eta = \sqrt{\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2} \end{aligned} \tag{69}$$

At the end of this section we would like to note that the noncommutative algebra of a canonical type is not rotationally invariant and that it is not invariant upon time reversal [70, 71, 65, 72–74]. Considering the transformations of coordinates and momenta after time reversal as $X_i \to X_i$, $P_i \to -P_i$ (similarly as in the ordinary space, $\theta = \eta = 0$) and taking into account the fact that the time reversal operation involves complex conjugation [18] after time reversal, it is concluded that the relations of the noncommutative algebra of a canonical type (49)–(51) transform to

$$[X_i, X_j] = -i\hbar\theta_{ij} \tag{70}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \sigma_{ij}) \tag{71}$$

$$[P_i, P_j] = -i\hbar\eta_{ij} \tag{72}$$

It follows from (70)–(72) that the relations (49)–(51) are not a time reversal invariant. Upon the time reversal the algebra (49)–(51) transforms into a noncommutative algebra with $-\theta_{ij}$, $-\eta_{ij}$.

For example, examining a simple problem of circular motion of a particle in a two-dimensional noncommutative phase space

$$[X_1, X_2] = i\hbar\theta \tag{73}$$

$$[X_i, P_j] = i\hbar\delta_{ij}(1 + \sigma) \tag{74}$$

$$[P_1, P_2] = i\hbar\eta \tag{75}$$

(θ, η, and σ are constants, $i, j = (1, 2)$) we find that this motion depends on its direction [74]. Namely, studying the Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - \frac{k}{\sqrt{X_1^2 + X_2^2}} \tag{76}$$

(here m is the mass of a particle, k is a constant) and taking into account the relations of noncommutative algebra which in the classical limit correspond to the following Poisson brackets

$$\{X_1, X_2\} = \theta \tag{77}$$

$$\{X_i, P_j\} = \delta_{ij}(1 + \sigma) \tag{78}$$

$$\{P_1, P_2\} = \eta \tag{79}$$

we find the following equations of motion

$$\dot{X}_1 = \frac{P_1}{m}(1+\sigma) + \frac{k\theta X_2}{X^3} \tag{80}$$

$$\begin{aligned}\dot{X}_2 &= \frac{P_2}{m}(1+\sigma) - \frac{k\theta X_1}{X^3}\\ \dot{P}_1 &= \frac{\eta P_2}{m} - \frac{kX_1}{X^3}(1+\sigma)\\ \dot{P}_2 &= -\frac{\eta P_1}{m} - \frac{kX_2}{X^3}(1+\sigma)\end{aligned} \tag{81}$$

$X = \sqrt{X_1^2 + X_2^2}$. The solution of (80)–(81) which corresponds to a circular motion with the radii R_0 reads

$$X_1(t) = R_0\cos(\omega t), \quad X_2(t) = R_0\sin(\omega t) \tag{82}$$

$$P_1(t) = -P_0\sin(\omega t), \quad P_2(t) = P_0\cos(\omega t) \tag{83}$$

where the momentum and the frequency are given by

$$P_0 = \frac{m\omega R_0^3 + km\theta}{R_0^2(1+\sigma)} \tag{84}$$

$$\omega = \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\sigma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} - \frac{\eta}{m} - \frac{k\theta}{R_0^3}\right) \tag{85}$$

In the case of a circular motion with the same radii R_0 in the opposite direction we have [74]

$$X_1(t) = R_0\cos(\omega t), \quad X_2(t) = -R_0\sin(\omega t) \tag{86}$$

$$P_1(t) = P_0'\sin(\omega t), \quad P_2(t) = P_0'\cos(\omega t) \tag{87}$$

$$P_0' = -\frac{m\omega' R_0^3 - km\theta}{R_0^2(1+\sigma)} \tag{88}$$

$$\omega' = \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}\left((1+\sigma)^2 - \theta\eta\right) + \left(\frac{k\theta}{R_0^3} + \frac{\eta}{m}\right)^2} + \frac{\eta}{m} + \frac{k\theta}{R_0^3}\right) \tag{89}$$

Note that the frequency of the circular motion depends on its direction. Note also that $P_0' \neq -P_0$ as it is in the ordinary space. The discrepancy in expressions (85), and (89) is an evident consequence of the time reversal symmetry breaking in the noncommutative phase space of a canonical type.

At this point it is worth mentioning that the expression for the frequency of the circular motion (89) can be obtained from (85) by changing θ to $-\theta$ and η to $-\eta$.

Note that because of the non-invariance of the noncommutative algebra (77)–(79) under the time reversal transformation one faces the problem of dependence of the transformation of noncommutative coordinates and noncommutative momenta upon the time reversal on their representation. The coordinates X_i, P_i which satisfy the relations of the noncommutative algebra (77)–(79) can be represented as

$$X_1 = \varepsilon(x_1 - \theta'_1 p_2) \tag{90}$$

$$X_2 = \varepsilon(x_2 + \theta'_2 p_1) \tag{91}$$

$$P_1 = \varepsilon(p_1 + \eta'_1 x_2) \tag{92}$$

$$P_2 = \varepsilon(p_2 - \eta'_2 x_1) \tag{93}$$

where coordinates and momenta x_i, p_i satisfy the ordinary commutation relations (15), (16); ε, θ'_1, θ'_2, η'_2, η'_2 are constants which satisfy the following relations

$$\varepsilon^2 = \frac{1}{1+\theta'_1\eta'_1} \tag{94}$$

$$\theta'_1\eta'_1 = \theta'_2\eta'_2 \tag{95}$$

$$\varepsilon^2(\theta'_1 + \theta'_2) = \theta \tag{96}$$

$$\varepsilon^2(\eta'_1 + \eta'_2) = \eta \tag{97}$$

Note that there are four equations (94)–(97) and five parameters ε, θ'_1, θ'_2, η'_1, η'_2. Therefore, there are different representations which corresponds to choosing one of the parameters ε, θ'_1, θ'_2, η'_2, η'_2. For example, if we consider $\theta'_2 = 0$, taking into account (94)–(97), we obtain $\varepsilon = 1$, $\eta'_1 = 0$, $\eta'_2 = \eta$, $\theta'_1 = \theta$ which correspond to the following representation

$$\begin{aligned} X_1 &= x_1 - \theta p_2 \\ X_2 &= x_2 \\ P_1 &= p_1 \\ P_2 &= p_2 - \eta x_1 \end{aligned} \tag{98}$$

In this case, upon the time reversal, considering the traditional transformations for coordinates and momenta $x_i \to x_i$, $p_i \to -p_i$ we find

$$\begin{aligned} X_1 &\to X'_1 = x_1 + \theta p_2 \\ X_2 &\to X_2 \end{aligned} \tag{99}$$

$$\begin{aligned} P_1 &\to -P_1 \\ P_2 &\to -P'_2 = -p_2 - \eta x_1 \end{aligned} \tag{100}$$

Note that according to (99), (100) the coordinate X_1, and the momentum P_2 do not transform traditionally.

Considering the parameters

$$\varepsilon = \frac{1}{\sqrt{1+\theta'\eta'}}$$
$$\theta'_1 = \theta'_2 = \frac{1 \pm \sqrt{1-\theta\eta}}{\eta} \tag{101}$$
$$\eta'_1 = \eta'_2 = \frac{1 \pm \sqrt{1-\theta\eta})}{\theta}$$

we obtain two symmetric representations [75, 76]

$$X_1 = \sqrt{\frac{\theta\eta}{2(1 \pm \sqrt{1-\theta\eta})}} \left(x_1 - \frac{1}{\eta}\left(1 \pm \sqrt{1-\theta\eta}\right) p_2 \right) \tag{102}$$

$$X_2 = \sqrt{\frac{\theta\eta}{2(1 \pm \sqrt{1-\theta\eta})}} \left(x_2 + \frac{1}{\eta}\left(1 \pm \sqrt{1-\theta\eta}\right) p_1 \right) \tag{103}$$

$$P_1 = \sqrt{\frac{\theta\eta}{2(1 \pm \sqrt{1-\theta\eta})}} \left(p_1 + \frac{1}{\theta}\left(1 \pm \sqrt{1-\theta\eta}\right) x_2 \right) \tag{104}$$

$$P_2 = \sqrt{\frac{\theta\eta}{2(1 \pm \sqrt{1-\theta\eta})}} \left(p_2 - \frac{1}{\theta}\left(1 \pm \sqrt{1-\theta\eta}\right) x_1 \right) \tag{105}$$

corresponding to the + or − signs in (102)–(105), and therefore, two different transformations of coordinates and momenta X_i, P_i upon the time reversal.

Thus, due to the non-invariance of the noncommutative algebra under the time reversal, the transformation of X_i, P_i depends on the choice of the parameters ε, θ'_1, θ'_2, η'_1, η'_2.

The way to preserve the time reversal symmetry in the noncommutative phase space (49)–(51) is to generalize the parameters of the noncommutativity to tensors which transform upon the time reversal as

$$\theta_{ij} \to -\theta_{ij} \tag{106}$$

$$\eta_{ij} \to -\eta_{ij} \tag{107}$$

In the next section we present a noncommutative algebra which is rotationally invariant, time reversal invariant and equivalent to a noncommutative algebra of a canonical type.

2.3. Rotationally and time reversal invariant noncommutative algebra of canonical type

In order to construct a noncommutative algebra which is invariant under the rotation and time reversal we consider the idea to generalize the parameters of the noncommutative algebra to tensors [77–79]. The tensors are considered to be constructed with the help of additional coordinates and additional momenta a_i, b_i, p^a_i, p^b_i. To preserve the rotational symmetry, these coordinates and momenta are assumed to be governed by rotationally symmetric systems. For the sake of simplicity these systems can be chosen to be harmonic oscillators

$$H^a_{osc}=\frac{(\mathbf{p}^a)^2}{2m_{osc}}+\frac{m_{osc}\omega^2_{osc}\mathbf{a}^2}{2} \tag{108}$$

$$H^b_{osc}=\frac{(\mathbf{p}^b)^2}{2m_{osc}}+\frac{m_{osc}\omega^2_{osc}\mathbf{b}^2}{2} \tag{109}$$

The lengths of the oscillators are considered to be equal to the Planck length

$$\sqrt{\frac{\hbar}{m_{osc}\omega_{osc}}}=l_P \tag{110}$$

and ω_{osc} is assumed to be very large (in this case oscillators which are in the ground states remain therein due to the large distance between the energy levels) [73, 80, 74].

For the sake of simplicity the tensors of noncommutativity which satisfy (106), (107) can be defined as [74]

$$\theta_{ij}=\frac{c_\theta}{\hbar}\sum_k \varepsilon_{ijk}p^a_k \tag{111}$$

$$\eta_{ij}=\frac{c_\eta}{\hbar}\sum_k \varepsilon_{ijk}p^b_k \tag{112}$$

where c_θ, c_η are constants.

Additional coordinates and momenta a_i, b_i, p^a_i, p^b_i are considered to satisfy the ordinary relations

$$[a_i,a_j]=[b_i,b_j]=[a_i,b_j]=[p^a_i,p^a_j]=[p^b_i,p^b_j]=[p^a_i,p^b_j]=0 \tag{113}$$

$$[a_i,p^a_j]=[b_i,p^b_j]=i\hbar\delta_{ij} \tag{114}$$

$$[a_i,p^b_j]=[b_i,p^a_j]=0 \tag{115}$$

besides

$$[a_i,X_j]=[a_i,P_j]=[p^a_i,X_j]=[p^a_i,P_j]=0 \tag{116}$$

$$[b_i,X_j]=[b_i,P_j]=[p^b_i,X_j]=[p^b_i,P_j]=0 \tag{117}$$

As a result, the commutation relations for coordinates and momenta have the following form [74]

$$[X_i,X_j]=ic_\theta\sum_k \varepsilon_{ijk}p^a_k \tag{118}$$

$$[X_i,P_j]=i\hbar\left(\delta_{ij}+\frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a\cdot\mathbf{p}^b)\delta_{ij}-\frac{c_\theta c_\eta}{4\hbar^2}p^a_j p^b_i\right) \tag{119}$$

$$[P_i, P_j] = ic_\eta \sum_k \varepsilon_{ijk} p_k^b \tag{120}$$

here we consider σ_{ij} to be defined as (54) which follows from the symmetric representation for coordinates and momenta satisfying (118), (120)

$$X_i = x_i + \frac{c_\theta}{2\hbar}[\mathbf{p}^a \times \mathbf{p}]_i \tag{121}$$

$$P_i = p_i + \frac{c_\eta}{2\hbar}[\mathbf{x} \times \mathbf{p}^b]_i \tag{122}$$

where for the operators x_i, p_i the ordinary relations hold (15), (16).

The algebra (118)–(120) is invariant upon time reversal, therefore, the transformations of the coordinates and momenta X_i, P_i upon time reversal do not depend on their representation and they read

$$X_i \to X_i, \quad P_i \to -P_i \tag{123}$$

Taking into account that upon the time reversal $x_i \to x_i$, $p_i \to -p_i$, $p_i^a \to -p_i^a$, $p_i^b \to -p_i^b$ from (121), (122) we obtain (123).

The operator of rotation in the space (118)–(120) is the following

$$\begin{aligned} U(\varphi) &= \exp\left(\frac{i\varphi(\mathbf{n}\cdot\mathbf{L^t})}{\hbar}\right) \\ \mathbf{L^t} &= [\mathbf{x}\times\mathbf{p}] + [\mathbf{a}\times\mathbf{p}^a] + [\mathbf{b}\times\mathbf{p}^b] \end{aligned} \tag{124}$$

here $\mathbf{n}$ is the unit vector, φ is an angle. Note that the operator $\mathbf{L^t}$ satisfies the following relations

$$\begin{aligned} [X_i, L_j^t] &= i\hbar\varepsilon_{ijk}X_k \\ [P_i, L_j^t] &= i\hbar\varepsilon_{ijk}P_k \\ [a_i, L_j^t] &= i\hbar\varepsilon_{ijk}a_k \\ [p_i^a, L_j^t] &= i\hbar\varepsilon_{ijk}p_k^a \\ [b_i, L_j^t] &= i\hbar\varepsilon_{ijk}b_k \\ [p_i^b, L_j^t] &= i\hbar\varepsilon_{ijk}p_k^b \end{aligned} \tag{125}$$

Also it commutes with operators $R = \sqrt{\sum_i X_i^2}$ and $P = \sqrt{\sum_i P_i^2}$

$$[L_i^t, R] = [L_i^t, P] = 0 \tag{126}$$

Thus, the distance and the absolute value of momentum remain the same after rotation [80]

$$\begin{aligned} R' &= U(\varphi)RU^+(\varphi) = R \\ P' &= U(\varphi)PU^+(\varphi) = P \end{aligned} \tag{127}$$

The algebra (118)–(120) is rotationally invariant, after rotation the commutation relations (118)–(120) read

$$[X'_i, X'_j] = ic_\theta \sum_k \varepsilon_{ijk} p_k^{a\prime} \tag{128}$$

$$[X'_i, P'_j] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar}(\mathbf{p}^{a\prime}\cdot\mathbf{p}^{b\prime})\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar} p_j^{a\prime} p_i^{b\prime}\right) \tag{129}$$

$$[P'_i, P'_j] = ic_\eta \sum_k \varepsilon_{ijk} p_k^{b\prime} \tag{130}$$

where

$$\begin{aligned} &X'_i = U(\varphi) X_i U^+(\varphi), \quad P'_i = U(\varphi) P_i U^+(\varphi) \\ &a'_i = U(\varphi) a_i U^+(\varphi), \quad p_i^{b\prime} = U(\varphi) p_i^b U^+(\varphi) \end{aligned} \tag{131}$$

$U^+(\varphi) = \exp(-i\varphi(\mathbf{n}\cdot\mathbf{L}^{\mathbf{t}})/\hbar)$.

Note that it follows from (116), (117) that the tensors of noncommutativity (579), (112) satisfy the following relations

$$\begin{aligned} &[\theta_{ij}, X_k] = [\theta_{ij}, P_k] = [\eta_{ij}, X_k] = [\eta_{ij}, P_k] = \\ &[\sigma_{ij}, X_k] = [\sigma_{ij}, P_k] = 0 \end{aligned} \tag{132}$$

which are the same as in the case of the noncommutative algebra of a canonical type (49)–(51) with θ_{ij}, η_{ij}, σ_{ij} being constants. Hence, the noncommutative algebra presented by (118)–(120) is invariant under rotations, time reversal transformations, besides, it is equivalent to the noncommutative algebra of a canonical type (49)–(51) [81].

2.4. Noncommutative algebra of Lie type

The noncommutative algebra of the Lie type is characterized by the following relations

$$[X_i, X_j] = i\hbar\theta_{ij}^k X_k \tag{133}$$

here θ_{ij}^k are antisymmetric to the lower index constants called parameters of noncommutativity. These constants can be chosen in particular cases (see, for instance, [82–92]).

In the particular case of the Lie-algebraic noncommutativity when the space coordinates commute to time, the relations of the noncommutative algebra are the following

$$[X_i, X_j] = \frac{i\hbar t}{\kappa}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right) \tag{134}$$

$$[X_i, P_j] = i\hbar\delta_{ij} \tag{135}$$

$$[P_i, P_j] = 0 \tag{136}$$

where the indexes ρ, τ are fixed and different, $i,j = (1,2,3)$, κ is a mass-like parameter [82, 93].

In the case when the space coordinates commute to space we have

$$[X_k,X_\gamma]=i\hbar\frac{X_l}{\tilde{\kappa}},\quad [X_l,X_\gamma]=-i\hbar\frac{X_k}{\tilde{\kappa}} \tag{137}$$

$$[P_k,X_\gamma]=i\hbar\frac{P_l}{\tilde{\kappa}},\quad [P_l,X_\gamma]=-i\hbar\frac{P_k}{\tilde{\kappa}} \tag{138}$$

$$[X_i,P_j]=i\hbar\delta_{ij},\quad [X_\gamma,P_\gamma]=i\hbar \tag{139}$$

$$[X_k,X_l]=[P_m,P_n]=0 \tag{140}$$

here $\tilde{\kappa}$ is a constant, k, l, γ are different and fixed, $k,l,\gamma=(1,2,3)$, $i\neq\gamma$, $j\neq\gamma$ and $m,n=(1,2,3)$, [82].

The generalized noncommutative algebra of the Lie type is characterized by the following relations

$$[X_i,X_j]=i\hbar(\theta^0_{ij}t+\theta^k_{ij}X_k) \tag{141}$$

$$[X_i,P_j]=i\hbar(\delta_{ij}+\bar{\theta}^k_{ij}X_k+\tilde{\theta}^k_{ij}P_k)$$

$$[P_i,P_j]=0 \tag{142}$$

where $i,j,k=(1,2,3)$, the parameters θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ are antisymmetric to the lower index parameters of noncommutativity, the Poisson brackets for time and the spatial coordinates are equal to zero [86]. Note that the parameters of noncommutativity θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ are constrained. The constrains are caused by the Jacobi identity. Taking into account these constrains in the paper [86] two types of noncommutative algebras of the Lie-type are determined. These are

$$[X_k,X_\gamma]=i\hbar\left(-\frac{t}{\kappa}+\frac{X_l}{\tilde{\kappa}}\right),\quad [X_l,X_\gamma]=i\hbar\left(\frac{t}{\kappa}-\frac{X_k}{\tilde{\kappa}}\right) \tag{143}$$

$$[X_k,X_l]=i\hbar\frac{t}{\kappa},\quad [P_k,X_\gamma]=i\hbar\frac{P_l}{\tilde{\kappa}} \tag{144}$$

$$[P_l,X_\gamma]=-i\hbar\frac{P_k}{\tilde{\kappa}},\quad [X_i,P_j]=i\hbar\delta_{ij} \tag{145}$$

$$[X_\gamma,P_\gamma]=i\hbar,\quad [P_m,P_n]=0 \tag{146}$$

and

$$[X_k,X_\gamma]=i\hbar\left(-\frac{t}{\kappa}+\frac{X_l}{\tilde{\kappa}}\right) \tag{147}$$

$$[X_l,X_\gamma]=i\hbar\left(\frac{t}{\kappa}-\frac{X_k}{\tilde{\kappa}}\right) \tag{148}$$

$$[P_k,X_\gamma]=i\hbar\left(\frac{X_l}{\bar{\kappa}}+\frac{P_l}{\tilde{\kappa}}\right) \tag{149}$$

$$[P_l,X_\gamma]=i\hbar\left(\frac{X_k}{\bar{\kappa}}-\frac{P_k}{\tilde{\kappa}}\right) \tag{150}$$

$$[X_i, P_j] = i\hbar\delta_{ij} \tag{151}$$

$$[X_\gamma, P_\gamma] = i\hbar \tag{152}$$

$$[X_k, X_l] = [P_m, P_n] = 0 \tag{153}$$

where the indexes k, l, γ are different and fixed. The algebra (143)–(146) can be obtained setting $\theta^0_{kl} = -\theta^0_{k\gamma} = 1/\kappa$, $\theta^0_{l\gamma} = 1/\kappa$, $\theta^l_{k\gamma} = -\theta^k_{l\gamma} = \tilde{\theta}^l_{k\gamma} = -\tilde{\theta}^k_{l\gamma} = 1/\tilde{\kappa}$. Algebra (148)–(153) corresponds to $\theta^0_{l\gamma} = -\theta^0_{k\gamma} = 1/\kappa$, $\theta^l_{k\gamma} = -\theta^k_{l\gamma} = 1/\tilde{\kappa}$, $\tilde{\theta}^l_{k\gamma} = -\tilde{\theta}^k_{l\gamma} = 1/\tilde{\kappa}$, $\bar{\theta}^l_{k\gamma} = -\bar{\theta}^k_{l\gamma} = 1/\bar{\kappa}$.

2.5. Relation between nonlinear and linear deformed algebras

Let us study a nonlinear deformed algebra characterized by the relation (8). The coordinates and momenta satisfying (8) can be represented as

$$P = p \tag{154}$$

$$X = if(p)\frac{d}{dp} \tag{155}$$

(here and throughout the section we consider $\hbar = 1$). The momentum representation (154) acts on the square integrable functions $\phi(p) \in \mathcal{L}^2(-a, a; f), (a \leq \infty)$. The norm is the following

$$\| \phi \|^2 = \int_{-a}^{a} \frac{dp}{f(p)} \, | \phi(p) |^2 \tag{156}$$

The operator X is Hermitian if $\phi(-a) = \phi(a)$ or $\phi(-a) = -\phi(a)$. Stronger boundary conditions $\phi(-a) = \phi(a) = 0$ were studied in [26].

Let us extend the algebra (8) considering an additional operator $F = f(p)$ [94]. Taking into account (154), (155) we obtain

$$\begin{aligned} &[X, F] = [if\frac{d}{dp}, f(p)] = iff' \\ &[P, F] = [p, f(p)] = 0 \end{aligned} \tag{157}$$

We require that the algebra of X, P and F should be linear. In order to close the algebra we consider

$$ff' = \alpha + \beta p + \gamma f \tag{158}$$

where α, β, γ are real parameters. Note that ff' is a function of p, therefore, the right-hand side of (158) $\alpha + \beta p + \gamma f$ does not contain X.

Taking into account $f(-p) = f(p)$ and (158) we can write

$$ff' = -\alpha + \beta p - \gamma f \tag{159}$$

On the basis of (158), (159) we obtain $\alpha=\gamma=0$. Therefore, we consider the following equation

$$ff'=\beta p \tag{160}$$

the solution of which reads

$$f(p)=\pm\sqrt{c+\beta p^2} \tag{161}$$

with c being the constant of integration. Choosing the sign "+" in (161) and $c=1$ we obtain the following deformation function

$$f(p)=\sqrt{1+\beta p^2} \tag{162}$$

In this case the algebra for X, P and F

$$[X,P]=iF \tag{163}$$
$$[X,F]=i\beta P \tag{164}$$
$$[P,F]=0 \tag{165}$$

is linear (this is an algebra of the Lie type). For this algebra the Casimir operator is the following

$$K=P^2-\frac{1}{\beta}F^2 \tag{166}$$

(the operator K commutes with all the elements of the algebra). Returning to the nonlinear deformed algebra we obtain the constant

$$K=p^2-\frac{1}{\beta}f^2(p)=-\frac{1}{\beta} \tag{167}$$

Hence, the nonlinear algebra (8) with a deformation function given by (162) can be transformed to the linear algebra (163)–(165) with three operators [94].

The operators satisfying the linear algebra (163)–(165) with $\beta=-\lambda^2$ can be represented as

$$X=\lambda\left(-ix\frac{\partial}{\partial y}+iy\frac{\partial}{\partial x}\right)=\lambda L_z \tag{168}$$
$$P=x \tag{169}$$
$$F=\lambda y \tag{170}$$

Taking into account (168)–(170) the Casimir operator (166) can be written as

$$K=p^2+\frac{1}{\lambda^2}F^2=x^2+y^2 \tag{171}$$

And in the nonlinear representations we have $K=1/\lambda^2$.

On the basis of (168)–(170) the linear algebra (163)–(165) with $\beta=-\lambda^2$ corresponds to the algebra of $L_z=xp_y-yp_z$, x, y.

At the end of this section it is interesting to note that the nonlinear algebra

$$[X,P]=i\sqrt{1-(\lambda_1^2X^2+\lambda_2^2P^2)} \tag{172}$$

is related to the Lie algebra for the angular momentum

$$[J_x,J_y]=iJ_z \tag{173}$$

$$[J_z,J_x]=iJ_y \tag{174}$$

$$[J_y,J_z]=iJ_x \tag{175}$$

The Casimir operator for (173)–(175) is the squared total angular momentum J^2. Considering a subspace with a fixed eigenvalue of the operator J^2

$$J^2=J_x^2+J_y^2+J_z^2=j(j+1) \tag{176}$$

(where $j=0,1,2,...$ or $j=1/2,3/2,5/2,...$) we can write

$$J_z=\pm\sqrt{j(j+1)-(J_x^2+J_y^2)} \tag{177}$$

Considering the subspace spanned by the eigenstates of J_z with positive eigenvalues, namely, choosing "+" in (177), we find

$$[J_x,J_y]=i\sqrt{j(j+1)-(J_x^2+J_y^2)} \tag{178}$$

Introducing the operators of position and momentum as

$$X=\lambda_2J_x \tag{179}$$

$$P=\lambda_1J_y \tag{180}$$

we find the following relation

$$\begin{gathered}[X,P]=i\lambda_1\lambda_2\sqrt{j(j+1)-\left(\frac{1}{\lambda_2^2}X^2+\frac{1}{\lambda_1^2}P^2\right)}= \\ i\sqrt{\lambda_1^2\lambda_2^2j(j+1)-\left(\lambda_1^2X^2+\lambda_2^2P^2\right)}\end{gathered} \tag{181}$$

which for

$$\lambda_1^2\lambda_2^2j(j+1)=1 \tag{182}$$

corresponds to the deformed algebra (172). Thus, the nonlinear deformed algebra (172) and the Lie algebra for the total angular momentum (173)–(175) are related.

Let us consider the operator

$$F = \lambda_1\lambda_2 J_z = \sqrt{1-(\lambda_1^2 X^2 + \lambda_2^2 P^2)} \tag{183}$$

Taking into account (179), (180), (183) we have

$$[X,P] = iF \tag{184}$$

$$[X,F] = -i\lambda_2^2 P \tag{185}$$

$$[P,F] = i\lambda_1^2 X \tag{186}$$

where the parameters λ_1 and λ_2 are related by (182). Therefore, introducing the operator (183) the nonlinear algebra (172) can be extended to the linear one (184)–(186).

Note that in the limit $\lambda_1 \to 0$ (the limit corresponds to the contraction procedure described in [95]) the algebra (184)–(186) related to the algebra of the angular momentum, corresponds to the algebra (163)–(165) related to the algebra of transformations in the Euclidian space. Also, in the limit $\lambda_1 \to 0$ the algebra (172) corresponds to (8) with the deformation function (162). Therefore, the contraction procedure relates two linear algebras and relates the corresponding nonlinear deformed algebras.

The Relation of the nonlinear algebra (172) and the Lie algebra for the total angular momentum (173)–(175) can be used to find the energy spectrum of the harmonic oscillator

$$H = \frac{1}{2}(P^2 + X^2) \tag{187}$$

in the space (172) with $\lambda_1 = \lambda_2 = \lambda$

$$[X,P] = i\sqrt{1-\lambda^2(X^2+P^2)} \tag{188}$$

If (182) is satisfied, the algebra (188) is related to the Lie algebra for the total angular momentum. Therefore,

$$\lambda^4 = \frac{1}{j(j+1)} \tag{189}$$

Taking into account (179), (180) the Hamiltonian reads

$$H = \frac{\lambda^2}{2}(J_x^2 + J_y^2) = \frac{\lambda^2}{2}(J^2 - J_z^2) \tag{190}$$

The operators J^2 and J_z have the eigenvalues $j(j+1)$, m, $-j \le m \le j$, respectively, $[J^2, J_z] = 0$. Thus, the eigenvalues of (190) read

$$E_m = \frac{1}{2\sqrt{j(j+1)}}(j(j+1) - m^2)$$

The maximal quantum number $m=j$ corresponds to the energy of the ground state. Rewriting $m=j-n$ with $n=0$ corresponding to the ground state energy, we have

$$E_n=\frac{1}{2\sqrt{j(j+1)}}(j(j+1)-(j-n)^2) \tag{191}$$

where $n=0,1,2,...,j$ if j is an integer and $n=0,1,...,j-1/2$ if j is a half integer [94]. In the limit $j\to\infty$ which corresponds to $\lambda\to 0$ from (191) we obtain the well known result for the harmonic oscillator energy levels in the ordinary space $E_n=n+1/2$.

3. Soccer-ball problem in deformed space with minimal length

Studies of physical systems in deformed space with a minimal length give a possibility to find effects of space quantization in their properties and to estimate the minimal length. In the papers [29, 96, 97] the perihelion shift of the Mercury planet has been examined in deformed space with a minimal length and the upper bound for the minimal length has been estimated. The authors of the papers [29, 97] faced a problem of an extremely small result for the minimal length which is much beyond the Planck length. It has been concluded that there is a problem of macroscopic bodies in deformed space with a minimal length which is similar to the problem of macroscopic bodies in Doubly Spatial Relativity and is known as the “soccer-ball problem” [98–101]. A composite system in the frame of a deformed algebra leading to a minimal length is examined in [45, 102–105]. In this chapter we show that the soccer-ball problem can be solved in deformed space with a minimal length due to the relation of the parameter of deformation and mass.

The chapter is organized as follows. In Section 3.1 we study the features of the description of the motion of a macroscopic body in one-dimensional deformed space (3). We show that the motion of the body is described by the effective parameter of deformation which is less than the parameters of deformation corresponding to the particles forming it. We also conclude that the properties of the kinetic energy are preserved, if we consider the parameter of deformation to be related to mass. In Section 3.2 it is shown that the same relation of the parameter of deformation with mass is important for recovering the weak equivalence principle in deformed space. A generalization of these results to the case of a three-dimensional deformed algebra leading to a minimal length is presented in Section 3.3. In Section 3.4 and Section 3.5 we find the Galilean and Lorentz transformation in deformed space. It is shown that these transformations do not depend on the mass of a particle (body), if the parameter of deformation is related to mass. In Section 3.6 the minimal length is estimated on the basis of studies of the perihelion shift of the Mercury planet. We conclude that extremely small results for the minimal length can be reexamined to a more relevant length, if we take into account features of the description of the motion of a macroscopic body in deformed space.

3.1. Soccer-ball problem in one-dimensional deformed space and properties of kinetic energy

Let us present the features of a description of the motion of a body in a deformed space with a minimal length. We start with the case when the space is characterized by the relation (3). In the classical limit from (3) we have the corresponding Poisson brackets

$$\{X,P\}=1+\beta P^2 \tag{192}$$

For a body of mass m described by the Hamiltonian

$$H=\frac{P^2}{2m} \tag{193}$$

taking into account (192), we obtain the following equations of motion

$$\dot{X}=\{X,H\}=\frac{P}{m}(1+\beta P^2) \tag{194}$$

$$\dot{P}=\{P,H\}=0$$

Note that due to (3) the relation between the momentum and the velocity is deformed. Up to the first order in β form (194) we have

$$P=m\dot{X}(1-\beta m^2\dot{X}^2)$$

Therefore, the Hamiltonian (193) (the kinetic energy of the body) can be rewritten as

$$H=\frac{m\dot{X}^2}{2}-\beta m^3\dot{X}^4 \tag{195}$$

On the other hand, the macroscopic body can be considered as a composite system made of N particles. Thus, let us study the case when a body is divided into N parts which can be considered as particles. These particles move with the same velocities as the whole body. The Hamiltonian of the body reads

$$H=\sum_a \frac{(P^{(a)})^2}{2m_a} \tag{196}$$

where index a is used to label the particles, $a=(1..N)$. Considering a general case when the coordinates and momenta of different particles satisfy the relation (3) with different parameters β_a, namely

$$\{X^{(a)},P^{(b)}\}=\delta_{ab}(1+\beta_a(P^{(a)})^2) \tag{197}$$

we obtain

$$\dot{X}_i^{(a)} = \frac{P_i^{(a)}}{m_a}(1+\beta_a(P^{(a)})^2) \tag{198}$$

Using (198), and taking into account the fact that the velocities of particles forming the body are the same and equal to the velocity of the body

$$\dot{X}_i^{(a)} = \dot{X}_i \tag{199}$$

the Hamiltonian (196) reads

$$H = \frac{m\dot{X}^2}{2} - \tilde{\beta} m^3 \dot{X}^4 \tag{200}$$

with

$$\tilde{\beta} = \sum_a \beta_a \mu_a^3 \tag{201}$$

$\mu_a = m_a/m$ and $m = \sum_a m_a$ [45].

Note that the expressions (195), (200) coincide, if the parameter of deformation corresponding to the macroscopic body is defined as (201) [102, 45]. Otherwise, the property of additivity of the kinetic energy is not preserved in the deformed space (3).

It is worth noting that for a body (a composite system) made of N particles with the same masses $m_a = m$ and parameters of deformation $\beta_a = \beta$, from (201) we find

$$\tilde{\beta} = \frac{\beta}{N^2} \tag{202}$$

Thus, there is a reduction in the effective parameter of deformation $\tilde{\beta}$ corresponding to the macroscopic body with respect to the parameters of deformation β corresponding to individual particles. The effect of a minimal length on the motion of macroscopic bodies is smaller than this effect on the elementary particles. This statement is naturally understandable and has to be taken into account when studying macroscopic bodies in the deformed space with a minimal length.

It is also worth noting that if we calculate the effective parameter of deformation for a composite system dividing it into two subsystems with the effective parameters of deformation

$$\begin{aligned} \tilde{\beta}_1 &= \sum_{a=1}^{N_1} \beta_a \left(\frac{m_a}{\sum_{b=1}^{N_1} m_b} \right)^3 \\ \tilde{\beta}_2 &= \sum_{a=N_1+1}^{N} \beta_a \left(\frac{m_a}{\sum_{b=N_1+1}^{N} m_b} \right)^3 \end{aligned} \tag{203}$$

(m_a are masses of particles forming the system, N_1 is the number of particles in the first subsystem) on the basis of (201) we find

$$\tilde{\beta}=\tilde{\beta}_1\left(\frac{\sum_{a=1}^{N_1}m_a}{\sum_{b=1}^{N}m_b}\right)^3+\tilde{\beta}_2\left(\frac{\sum_{a=N_1+1}^{N}m_a}{\sum_{b=1}^{N}m_b}\right)^3=\frac{\sum_{a=1}^{N}\beta_a m_a^3}{\sum_{b=1}^{N}m_b} \tag{204}$$

It corresponds to the initial definition of the effective parameter of deformation for a composite system made of N particles (201) [45].

Let us also consider the property of independence of the kinetic energy on the composition. To examine this property in the deformed space with the minimal length (3) it is enough to consider a body with mass m which consists of two parts which can be treated as particles. The masses and parameters of deformation of these particles are $m_1=m\mu$ and $m_2=m(1-\mu)$, $0\leq\mu\leq 1$, $\beta_1=\beta_\mu$ and $\beta_2=\beta_{1-\mu}$. In this case the effective parameter of deformation reads

$$\tilde{\beta}=\beta_\mu\mu^3+\beta_{1-\mu}(1-\mu)^3 \tag{205}$$

The kinetic energy does not depend on the composition, if the parameter $\tilde{\beta}$ which corresponds to the body does not depend on its composition. Namely the parameter $\tilde{\beta}$ has to be the same for different μ. Therefore, the equation (205) can be considered as the equation for β_μ for a given $\tilde{\beta}$. The solution of this equation reads

$$\beta_\mu=\frac{\tilde{\beta}}{\mu^2} \tag{206}$$

Using $\mu=m_1/m$, we can rewrite (206) as

$$\beta_1 m_1^2=\tilde{\beta}m^2 \tag{207}$$

Thus, in order to recover the independence of the kinetic energy of the composition in the deformed space, the product $\sqrt{\beta}_a m_a$ has to be the same for different particles

$$\sqrt{\beta}_a m_a=\gamma=const \tag{208}$$

where γ is a constant which does not depend on mass ($1/\gamma$ has the dimension of velocity) [102]. Therefore, we have that the following relation is satisfied for the parameter of deformation corresponding to a particle and the parameter of deformation corresponding to the body (201)

$$\sqrt{\beta}_a m_a=\sqrt{\tilde{\beta}}m=\gamma \tag{209}$$

Note that due to the relation (208) the properties of the kinetic energy are preserved in all orders in the parameter of deformation [106]. If (208) is satisfied, the momentum is proportional to mass as it is in the ordinary space. Namely, taking into account (194), (208) we can write

$$\dot{X} = \frac{P}{m}\left(1+\gamma\frac{P^2}{m^2}\right) \tag{210}$$

From (210), the ratio P/m is a function of $\dot{X}$ and γ, and it is independent of mass

$$\frac{P}{m} = g(\dot{X},\gamma) \tag{211}$$

Therefore, the momentum P is proportional to mass m, $P = mg(\dot{X},\gamma)$. As a result the kinetic energy of a body with mass m can be rewritten as

$$H = \frac{P^2}{2m} = \frac{m(g(\dot{X},\gamma))^2}{2} \tag{212}$$

For a body which can be divided into N parts (particles) with masses m_a according to the additivity property we can write

$$H = \sum_a H_a = \sum_a \frac{m_a(g(\dot{X},\gamma))^2}{2} = \frac{m(g(\dot{X},\gamma))^2}{2} \tag{213}$$

where $m = \sum_a m_a$ is the mass of the body. From (212), (213) we have that the additivity property of the kinetic energy is satisfied. Note also that the kinetic energy of a body (212), (213) depends on its mass m and constant γ and does not depend on the composition. Hence, if the relation (208) holds, the properties of the kinetic energy are preserved in all orders in the parameters of deformation.

Note that this conclusion can be generalized to the case of the deformed algebra (8) with the arbitrary deformation function $f(P)$. Taking into account (8), (193) we have

$$\dot{X} = \frac{P}{m}f(P) \tag{214}$$

Note that it follows from the dimensional considerations that the function $f(P)$ in (8) has to be dimensionless. Therefore, we can write

$$f(P) = \tilde{f}(\sqrt{\beta}P) \tag{215}$$

Hence, if the relation (208) holds, (214) can be rewritten as

$$\dot{X} = \frac{P}{m}\tilde{f}\left(\gamma\frac{P}{m}\right) \tag{216}$$

It follows from (216) that the ratio P/m depends on $\dot{X}$ and γ (211), therefore, the momentum is proportional to mass and we can write (212), (213). Thus, the properties of the kinetic energy are preserved [106].

The relation (208) is also important for recovering the weak equivalence principle in the deformed space (3). This is shown in the next section.

3.2. Free-fall of a particle in uniform gravitational field in deformed space and the weak equivalence principle

The Hamiltonian of a particle with mass m in a uniform field reads

$$H=\frac{P^2}{2m}-mgX \tag{217}$$

g is a constant which characterizes the field. Note that in the Hamiltonian we consider the inertial mass of the particle (the first term in (217)) to be equal to the gravitational mass (the second term in (217)).

Taking into account (192), the equations of motion of the particle are the following

$$\dot{X}=\{X,H\}=\frac{P}{m}(1+\beta P^2) \tag{218}$$

$$\dot{P}=\{P,H\}=mg(1+\beta P^2) \tag{219}$$

Considering the zero initial conditions $X(0)=0$, and $P(0)=0$, from (218), (219) we obtain

$$X=\frac{1}{2gm^2\beta}\tan^2(\sqrt{\beta}mgt) \tag{220}$$

$$\dot{X}=\frac{1}{m\sqrt{\beta}}\frac{\tan(\sqrt{\beta}mgt)}{\cos^2(\sqrt{\beta}mgt)} \tag{221}$$

Note that the motion is periodic [102]. The particle moves from $X=0$ to $X=\infty$, then reflects from ∞ and moves in the opposite direction to $X=0$. The period of the motion is given by

$$T=\frac{\pi}{m\sqrt{\beta}g} \tag{222}$$

The solution (220) is correct for $t\ll T$ which corresponds to the nonrelativistic case (the velocity of the particle is much smaller than the speed of light) [102].

In the first order in the parameter of deformation we have

$$\dot{X}=gt\left(1+\frac{4}{3}\beta m^2g^2t^2\right) \tag{223}$$

$$X=\frac{gt^2}{2}\left(1+\frac{2}{3}\beta m^2g^2t^2\right) \tag{224}$$

Note that for $\beta\to 0$ the expressions (223), (224) reduce to well the known result for a free-falling particle $\dot{X}=gt$, $X=gt^2/2$.

It is important to mention the weak equivalence principle that in the deformed space, also known as the universality of free fall or the Galilean equivalence principle, is not satisfied. According to this principle the velocity and the position of the particle in a gravitational field do not depend on its composition and mass. Note that the velocity and the position of a free-falling particle (223), (224) depend on its mass m, namely they depend on the product βm^2, therefore, the weak equivalence principle is violated [102, 107].

Besides, for a free-falling body with mass m, taking into account that its motion is described by the effective parameter of deformation (201) on the basis of (224) we have the following trajectory

$$X = \frac{gt^2}{2}\left(1 + \frac{2}{3}\tilde{\beta} m^2 g^2 t^2\right) \tag{225}$$

where $\tilde{\beta}$ is given by (201) [102]. Note that $\tilde{\beta}$ depends on the masses of particles forming the body, and therefore, depends on its composition. This also violates the equivalence principle. The trajectories of bodies in a uniform gravitational field with the same masses but different compositions are different.

Note that the deformation (192) causes a great violation of the weak equivalence principle. Taking into account (224) the accelerations of bodies with masses m_1, m_2 read

$$\begin{aligned} \ddot{X}^{(1)} &= g + 4\beta m_1^2 g^2 t^2 \\ \ddot{X}^{(2)} &= g + 4\beta m_2^2 g^2 t^2 \end{aligned} \tag{226}$$

Therefore, up to the first order in β the Eötvös parameter for these particles is the following

$$\frac{\Delta a}{a} = \frac{2(\ddot{X}^{(1)} - \ddot{X}^{(2)})}{\ddot{X}^{(1)} + \ddot{X}^{(2)}} = 4\upsilon^2 \beta (m_1^2 - m_2^2) \tag{227}$$

where υ is the velocity of the free-falling particle in the ordinary space ($\beta = 0$). Considering $\hbar\sqrt{\beta}$ to be equal to the Planck length,

$$\hbar\sqrt{\beta} = l_P = \sqrt{\frac{\hbar G}{c^3}} \tag{228}$$

the expression (227) can be rewritten as

$$\frac{\Delta a}{a} = 4\frac{\upsilon^2}{c^2}\frac{(m_1^2 - m_2^2)}{m_P^2} \tag{229}$$

where $m_P = \sqrt{\hbar c/G}$ is the Planck mass, c is the speed of light. Note that for bodies with masses $m_1 = 1$ kg, $m_2 = 0.1$ kg, considering $\upsilon = 1$ m/s we obtain a great violation of the equivalence principle [106]

$$\frac{\Delta a}{a} \approx 0.1 \tag{230}$$

which has to be observed experimentally.

It is worth mentioning that tests of the weak equivalence principle show that this principle holds with the high accuracy. According to the Lunar Laser ranging experiment results $\Delta a/a=(-0.8\pm1.3)\cdot10^{-13}$ [108], laboratory torsion-balance tests give similar limits on the violation of the weak equivalence principle, namely $\Delta a/a=(-0.7\pm1.3)\cdot10^{-13}$ for Be and Al, and $\Delta a/a=(0.3\pm1.8)\cdot10^{-13}$ for Be and Ti [109]. The MICROSCOPE space mission aims to test the validity of the this principle at the level 10^{-15} [110].

It is important to stress that if we assume that the relation (208) from which (209) follows is satisfied, for a body (a particle) in a uniform field, we have the following trajectory

$$X=\frac{gt^2}{2}\left(1+\frac{2}{3}\gamma^2g^2t^2\right) \tag{231}$$

which depends on the constant γ (this constant is the same for different bodies (particles)) and does not depend on its mass and composition. Hence, the weak equivalence principle is recovered due to the relation (208).

Note that in the general case of the deformed algebra (8) and a nonuniform gravitational field $V(X)$ for a particle (body) with mass m we have

$$H=\frac{P^2}{2m}+mV(X) \tag{232}$$

and the equation of motion reads

$$\dot{X}=\frac{P}{m}\tilde{f}(\sqrt{\beta}P) \tag{233}$$

$$\dot{P}=-m\frac{\partial V}{\partial X}\tilde{f}(\sqrt{\beta}P) \tag{234}$$

Here, we take into account (215). If the relation (208) is satisfied we can write

$$\dot{X}=\frac{P}{m}\tilde{f}\left(\gamma\frac{P}{m}\right) \tag{235}$$

$$\frac{\dot{P}}{m}=-\frac{\partial V}{\partial X}\tilde{f}\left(\gamma\frac{P}{m}\right) \tag{236}$$

The solutions of the equations (235), (236) $X(t)$ and $P(t)/m$ do not depend on mass. Hence, in the case of an arbitrary deformation function (8) the motion of a particle (body) in a gravitational field does not depend on its mass due to the relation (208) [106].

At the end of this section let us estimate the value of the constant γ in (208). This constant is the same for particles with different masses and has a dimension inverse to velocity. Therefore, let us introduce a dimensionless constant γc, with c being the speed of light. Assuming that for an electron, the minimal length $\hbar\sqrt{\beta}_e$ is related to the Planck length

$$\hbar\sqrt{\beta}_e=l_P=\sqrt{\hbar G/c^3} \tag{237}$$

we find

$$\gamma c = c\sqrt{\beta} m_e = \sqrt{\alpha \frac{Gm_e^2}{e^2}} \simeq 4.2\times 10^{-23} \tag{238}$$

α is the fine structure constant, $\alpha = e^2/\hbar c$ [102].

Note that if we fix the parameter β_e corresponding to the electron as (237) for parameters of deformation corresponding to other particles we find

$$\hbar\sqrt{\beta} = \frac{m_e}{m}\hbar\sqrt{\beta}_e = \frac{m_e}{m} l_P \tag{239}$$

where β corresponds to the particle with mass m. For instance for nucleons it is found that the effective minimal length is three orders smaller than the minimal length for electrons

$$\hbar\sqrt{\beta}_{\text{nuc}} \simeq \frac{l_P}{1840} \tag{240}$$

In the next section we generalize the obtained results to the case of a three-dimensional deformed space (10)–(12).

3.3. Motion of a body in three-dimensional deformed space with minimal length and weak equivalence principle

In this section we present the features of the description of the motion of a body in a three-dimensional space with a minimal length which is characterized by the relations (10)–(12). In the classical limit from (10)–(12) we have the following deformed Poisson brackets

$$\{X_i, X_j\} = \frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1+\beta P^2}(P_i X_j - P_j X_i) \tag{241}$$

$$\{X_i, P_j\} = \delta_{ij}(1+\beta P^2) + \beta' P_i P_j \tag{242}$$

$$\{P_i, P_j\} = 0 \tag{243}$$

For a body of mass m with the Hamiltonian

$$H = \frac{P^2}{2m} \tag{244}$$

(where $P^2 = \sum_i P_i^2$) in the deformed space (241)–(243) the equations of motion read

$$\dot{X}_i = \frac{P_i}{m}(1+(\beta+\beta')P^2) \tag{245}$$

$$\dot{P}_i = 0 \tag{246}$$

Using (245) up to the first order in β and β' we can write

$$P_i = \frac{m\dot{X}_i}{1+(\beta+\beta')m^2\dot{X}^2} \tag{247}$$

with $\dot{X}^2=\sum_i \dot{X}_i^2$. Therefore, the Hamiltonian can be rewritten as

$$H=\frac{m\dot{X}^2}{2}(1-2(\beta+\beta')m^2\dot{X}^2) \tag{248}$$

Diversely, considering the macroscopic body as a composite system made of N particles with masses m_a which move with the same velocities we can write the following Hamiltonian

$$H=\sum_a \frac{(P^{(a)})^2}{2m_a} \tag{249}$$

Relations of the deformed algebra (241)–(243) can be generalized as

$$\{X_i^{(a)},X_j^{(b)}\}=\delta_{ab}\frac{(2\beta_a-\beta'_a)+(2\beta_a+\beta'_a)\beta_a(P^{(a)})^2}{1+\beta_a(P^{(a)})^2}\times$$
$$(P_i^{(a)}X_j^{(a)}-P_j^{(a)}X_i^{(a)}) \tag{250}$$
$$\{X_i^{(a)},P_j^{(b)}\}=\delta_{ab}\delta_{ij}(1+\beta_a(P^{(a)})^2)+\delta_{ab}\beta'_aP_i^{(a)}P_j^{(a)} \tag{251}$$
$$\{P_i^{(a)},P_j^{(a)}\}=0 \tag{252}$$

where $X_i^{(a)}$, $P_i^{(a)}$ are the coordinates and momenta of a particle with index a. Therefore, the equations of motion read

$$\dot{X}_i^{(a)}=\frac{P_i^{(a)}}{m_a}(1+(\beta_a+\beta'_a)(P^{(a)})^2) \tag{253}$$
$$\dot{P}_i^{(a)}=0 \tag{254}$$

From (253), taking into account that the velocities of the particles are the same $\dot{X}_i^{(a)}=\dot{X}_i$ up to the first order in the parameters of deformation the Hamiltonian (249) can be rewritten as

$$H=\frac{m\dot{X}^2}{2}(1-2m^2(\tilde{\beta}+\tilde{\beta}')\dot{X}^2) \tag{255}$$

here $\tilde{\beta}$, $\tilde{\beta}'$ are effective parameters of deformation given by (201) and

$$\tilde{\beta}'=\sum_a \beta'_a\mu_a^3 \tag{256}$$

$\mu_a=m_a/m$, $m=\sum_a m_a$.

Note that it follows from (255) that the kinetic energy of the body depends on $\tilde{\beta}$, $\tilde{\beta}'$ which depend on the composition of the body. Also the expressions (248), (255) do not coincide. Therefore, the additivity property of the kinetic energy is not preserved.

Similarly as in the one-dimensional case, the kinetic energy properties are preserved, if the parameters of deformation are related to mass such as

$$\sqrt{\beta_a}m_a = \sqrt{\tilde{\beta}}m = \gamma = const \tag{257}$$

$$\sqrt{\beta'_a}m_a = \sqrt{\tilde{\beta}'}m = \gamma' = const \tag{258}$$

where γ, γ' are constants which are the same for different particles, β_a, β'_a are parameters of deformation which corresponds to a particle with mass m_a, $\tilde{\beta}$, $\tilde{\beta}'$ are effective parameters of deformation which correspond to a body with mass m [105, 111].

It is important to note another result which can be obtained if the relations (257), (258) are satisfied. Namely, due to the relations (257), (258) the Poisson brackets for coordinates and momenta of the center-of-mass of a body

$$\tilde{\mathbf{X}} = \sum_a \mu_a \mathbf{X}^{(a)} \tag{259}$$

$$\tilde{\mathbf{P}} = \sum_a \mathbf{P}^{(a)} \tag{260}$$

reproduce the relations of the deformed algebra [105]

$$\{\tilde{X}_i, \tilde{X}_j\} = \sum_a \mu_a^2 \frac{(2\beta_a - \beta'_a) + (2\beta_a + \beta'_a)\beta_a (P^{(a)})^2}{1 + \beta_a (P^{(a)})^2}(P_i^{(a)} X_j^{(a)} - P_j^{(a)} X_i^{(a)}) =$$
$$\frac{(2\tilde{\beta} - \tilde{\beta}') + (2\tilde{\beta} + \tilde{\beta}')\tilde{\beta}\tilde{P}^2}{1 + \tilde{\beta}\tilde{P}^2}(\tilde{P}_i \tilde{X}_j - \tilde{P}_j \tilde{X}_i) \tag{261}$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \sum_a \mu_a \delta_{ij}(1 + \beta_a (P^{(a)})^2) + \sum_a \mu_a \beta'_a P_i^{(a)} P_j^{(a)} =$$
$$\delta_{ij}(1 + \tilde{\beta}\tilde{P}^2) + \tilde{\beta}' \tilde{P}_i \tilde{P}_j \tag{262}$$

$$\{\tilde{P}_i, \tilde{P}_j\} = 0 \tag{263}$$

with the effective parameters $\tilde{\beta}$ and $\tilde{\beta}'$

$$\tilde{\beta} = \frac{\gamma^2}{m^2} \tag{264}$$

$$\tilde{\beta}' = \frac{(\gamma')^2}{m^2} \tag{265}$$

Writing (261)–(263) we take into account that if the relations (257), (258) hold the ratio $P_i^{(a)}/m_a$ is the same for particles which move with the same velocities.

Namely, for particles which form the body and move with the same velocities we have that $P_i^{(a)}/m_a$ depends on $\dot{X}_i$, γ, γ'

$$P_i^{(a)\prime}\left(1+(\gamma^2+(\gamma')^2)(P_i^{(a)\prime})^2\right)=\dot{X}_i \tag{266}$$

$$P_i^{(a)\prime}=\frac{P_i^{(a)}}{m_a} \tag{267}$$

Therefore, we can write

$$P_i^{(a)}=\frac{m_a}{m}\tilde{P}_i \tag{268}$$

Also due to the conditions (257), (258) the weak equivalence principle is recovered in the deformed space (241)–(243) [105]. For a particle of mass m in a gravitational field with the Hamiltonian

$$H=\frac{P^2}{2m}+mV(X_1,X_2,X_3) \tag{269}$$

(function $V(X_1,X_2,X_3)$ describes the field), considering the deformed Poisson brackets (241)–(243), we have the following equations of motion

$$\begin{gathered}\dot{X}_i=\frac{P_i}{m}\left(1+(\beta+\beta')P^2\right)+\\ m\frac{(2\beta-\beta')+(2\beta+\beta')\beta P^2}{1+\beta P^2}(P_iX_j-P_jX_i)\frac{\partial V}{\partial X_j}\end{gathered} \tag{270}$$

$$\dot{P}_i=-m\left(1+\beta P^2\right)\frac{\partial V}{\partial X_i}-m\beta' P_iP_j\frac{\partial V}{\partial X_j} \tag{271}$$

The equations of motion of the particle in the gravitational field depend on its mass, hence, deformation leads to the violation of the weak equivalence principle. Note that considering the conditions (257), (258) we can write

$$\begin{gathered}\dot{X}_i=P_i'\left(1+(\gamma+\gamma')(P')^2\right)+\\ \frac{(2\gamma-\gamma')+(2\gamma+\gamma')\gamma(P')^2}{1+\gamma(P')^2}(P_i'X_j-P_j'X_i)\frac{\partial V}{\partial X_j}\end{gathered} \tag{272}$$

$$\dot{P}_i'=-\left(1+\gamma(P')^2\right)\frac{\partial V}{\partial X_i}-\gamma' P_i'P_j'\frac{\partial V}{\partial X_j} \tag{273}$$

with $P_i'=P_i/m$. The solutions of equations (272), (273) $X_i(t)$, $P_i'(t)$ do not depend on mass. Therefore, the weak equivalence principle is recovered.

In the case of the motion of a body with mass m in the gravitational field we have $H=\tilde{P}^2/2m+mV(\tilde{X}_1,\tilde{X}_2,\tilde{X}_3)$. If the parameters of deformation are determined by mass such as (257), (258) the coordinates and momenta of the center-of-mass $\tilde{X}_i$, $\tilde{P}_i$ satisfy the relations (261)–(263) and the equations of motion read

$$\dot{\tilde{X}}_i = \tilde{P}'_i\left(1+(\gamma+\gamma')(\tilde{P}')^2\right)+ \frac{(2\gamma-\gamma')+(2\gamma+\gamma')\gamma(\tilde{P}')^2}{1+\gamma(\tilde{P}')^2}(\tilde{P}'_i\tilde{X}_j-\tilde{P}'_j\tilde{X}_i)\frac{\partial V}{\partial \tilde{X}_j} \tag{274}$$

$$\dot{\tilde{P}}'_i = -\left(1+\gamma(\tilde{P}')^2\right)\frac{\partial V}{\partial \tilde{X}_i} - \gamma'\tilde{P}'_i\tilde{P}'_j\frac{\partial V}{\partial \tilde{X}_j} \tag{275}$$

Here, the following notation $\tilde{P}'_i = \tilde{P}_i/m$ is used. The equations (274)–(275) and their solutions do not depend on the mass of the body and on its composition. Therefore, the weak equivalence principle is preserved [105].

In the next sections we show that the relation of the parameter of deformation with mass is also important for providing the independence of the Galilean and Lorentz transformations of mass.

3.4. Galilean transformation in deformed space and parameters of deformation

In the first order in β the Galilean transformations in deformed space are similar to the Lorentz transformations [43]. Let us show this in details.

The Hamiltonian of a particle of mass m in the potential $U(X)$ moving in deformed space (3) written in the representation (4), (5) is the following

$$H = \frac{P^2}{2m} + U(X) = \frac{\tan^2(\sqrt{\beta}p)}{2m\beta} + U(x) \tag{276}$$

Up to the first order in β we have

$$H = \frac{p^2}{2m} + \frac{1}{3}\frac{\beta}{m}p^4 + U(x) \tag{277}$$

Note that the expression (277) is similar to the relativistic Hamiltonian written up to the first order in $1/c^2$

$$H_r = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} + U(x) = mc^2 + \frac{p^2}{2m} - \frac{1}{8m^3c^2}p^4 + U(x) \tag{278}$$

Namely, the Hamiltonian (277) can be obtained from the following equation

$$H = -mu^2\sqrt{1-\frac{p^2}{m^2u^2}} + mu^2 + U(x) \tag{279}$$

in the first order in β or in the first order in $1/u^2$ where

$$u^2 = \frac{3}{8\beta m^2} \tag{280}$$

Hence, all properties related to deformations are similar to the relativistic properties with an opposite sign before $1/c^2$.

In the first order in β the Galilean transformations are similar to the Lorentz transformations with an opposite sign before $1/c^2$. To show this let us find the Lagrangian of a classical particle in space with a minimal length starting from the Hamiltonian formalism. We have the following expression for $\dot{x}$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} + \frac{4}{3}\frac{\beta}{m}p^3 \tag{281}$$

from which it follows that in the first order in β the momentum as a function of x, $\dot{x}$ reads

$$p = m\dot{x}\left(1 - \frac{4}{3}\beta m^2\dot{x}^2\right) \tag{282}$$

Therefore, for the Lagrangian we have

$$L = \dot{x}p - H(x,p) = \frac{m\dot{x}^2}{2} - \frac{1}{3}\beta m^3\dot{x}^4 - U(x) \tag{283}$$

Note that the Lagrangian is very similar to the Lagrangian of a relativistic particle written in the first order over $1/c^2$

$$L_r = -mc^2\sqrt{1 - \frac{\dot{x}^2}{c^2}} - U(x) = -mc^2 + \frac{m\dot{x}^2}{2} + \frac{m}{8c^2}\dot{x}^4 - U(x) \tag{284}$$

Namely, we have

$$L = mu^2\sqrt{1 + \frac{\dot{x}^2}{u^2}} - mu^2 - U(x) \tag{285}$$

where u is the effective velocity (280). The constant $-mu^2$ can be omitted because it has no influence on the equations of motion. We would like to note that the Lagrangian (285) corresponds to (283) in the first order over $1/u^2$ (or in the first order over β).

For a free particle we have the following Lagrangian

$$L = mu^2\sqrt{1 + \frac{\dot{x}^2}{u^2}} \tag{286}$$

Thus, in the first order in β the action can be written as

$$S = mu^2\int_{t_1}^{t_2}\sqrt{1 + \frac{\dot{x}^2}{u^2}}dt = mu^2\int_{(1)}^{(2)} ds \tag{287}$$

$$ds^2 = u^2(dt)^2 + (dx)^2 \tag{288}$$

The interval (288) is invariant under rotation in the plane (ut, x). Therefore, the symmetry transformations are the following

$$
\begin{aligned}
x &= x' \cos\phi + ut' \sin\phi \\
ut &= -x' \sin\phi + ut' \cos\phi
\end{aligned}
\tag{289}
$$

The angle ϕ is related to the velocity V of motion of the point $x' = 0$ with respect to the rest of the frame of reference

$$
\frac{V}{u} = \frac{x}{ut} = \tan\phi \tag{290}
$$

Hence, the Galilean transformation in the deformed space reads [43]

$$
x = \frac{x' + Vt'}{\sqrt{1 + V^2/u^2}} \tag{291}
$$

$$
t = \frac{t' - x'V/u^2}{\sqrt{1 + V^2/u^2}} \tag{292}
$$

Note that the transformation (291), (292) corresponds to the Lorenz transformation with $1/c^2$ changed to $-1/u^2$. It is worth mentioning that this transformation is correct only in the first order over β (the parameter β is related to $1/u^2$, see (280)). Therefore, in the first order over the parameter of deformation we can write

$$
x = (x' + Vt')\left(1 - \frac{V^2}{2u^2}\right) \tag{293}
$$

$$
t = t'\left(1 - \frac{V^2}{2u^2}\right) - x'\frac{V}{u^2} \tag{294}
$$

In the limit $\beta \to 0$ $(u \to \infty)$ the transformation (293), (294) corresponds to the ordinary Galilean transformation.

It is worth mentioning that the effective velocity (280) depends on mass, therefore, the Galilean transformation in the deformed space depends on the mass of a particle. It is important to stress that if the relation (208) is satisfied we have

$$
u^2 = \frac{3}{8\gamma^2} \tag{295}
$$

Therefore, due to the relation of the parameter of deformation with mass (208) the Galilean transformation is the same for coordinates of different particles as everybody feels it must be.

The result can be generalized to the three-dimensional case. Let us consider the three-dimensional deformed algebra (27), (28) which is invariant with respect to the translations in the configuration space. In the classical limit we obtain the following Poisson brackets

$$\begin{aligned}\{X_i,P_j\} &= \sqrt{1+\beta P^2}\left(\delta_{ij}+\beta P_i P_j\right)\\ \{X_i,X_j\} &= \{P_i,P_j\}=0\end{aligned} \tag{296}$$

Considering the representation (29), (30) up to the first order in the parameter of deformation we can write the following Hamiltonian

$$H=\frac{P^2}{2m}+U(\mathbf{X})=\frac{1}{2m}\frac{p^2}{1-\beta p^2}+U(\mathbf{x})=\frac{p^2}{2m}+\frac{\beta}{2m}p^4+U(\mathbf{x}) \tag{297}$$

The velocity reads

$$\dot{x}_i=\frac{p_i}{m}(1+2\beta p^2) \tag{298}$$

therefore, we find

$$p_i=m\dot{x}_i(1-2\beta\dot{x}^2) \tag{299}$$

Hence, the Lagrangian has the following form

$$L=\frac{m\dot{\mathbf{x}}^2}{2}-\frac{\beta m^3}{2}\dot{\mathbf{x}}^4-U(\mathbf{x}) \tag{300}$$

In the first order over β the Lagrangian (300) can be written as (286). The action is (287) with

$$ds^2=u^2(dt)^2+(dx_1)^2+(dx_2)^2+(dx_3)^2 \tag{301}$$

$$u^2=\frac{1}{4\beta m^2}=\frac{1}{4\gamma^2} \tag{302}$$

Thus, if the frame of reference $(t',\mathbf{x}')$ moves along axis x_1 with respect to another frame of reference $(t,\mathbf{x})$ with velocity V the Galilean transformation of the coordinate x_1' and time t' to the coordinate x_1 and time t reads (293), (294), and for other coordinates we have $x_2=x_2'$, $x_3=x_3'$.

At the end of this section let us estimate the value of the effective velocity (302). For this purpose we consider the result for the constant γ (238) and obtain $u\simeq 1.2\times 10^{22}c$ [43].

In the next section the obtained results are generalized for the relativistic case.

3.5. Lorentz transformation in deformed space

The one-dimensional relativistic Hamiltonian of a free particle reads

$$H=mc^2\sqrt{1+\frac{P^2}{m^2c^2}} \tag{303}$$

Considering the representation (4), (5) in the first order over β and $1/c^2$ we have

$$H = m^2c^2 + \frac{p^2}{2m} - \left(\frac{1}{8m^2c^2} - \frac{\beta}{3}\right)\frac{p^4}{m} \tag{304}$$

It is convenient to introduce the following notation

$$\frac{1}{8m^2\tilde{c}^2} = \frac{1}{8m^2c^2} - \frac{\beta}{3} \tag{305}$$

Note that in the first order over $1/\tilde{c}^2$ from

$$H = m\tilde{c}^2\sqrt{1 + \frac{p^2}{m^2\tilde{c}^2}} - m\tilde{c}^2 + mc^2 \tag{306}$$

we obtain (304). Assuming that $\beta \ll 1/m^2c^2$ we have that the Hamiltonian corresponds to the relativistic Hamiltonian with the effective velocity $\tilde{c}$. Note that $\tilde{c} > c$ and $\tilde{c} \to c$ when $\beta \to 0$.

Therefore, the Lorentz transformation in the deformed space corresponds to the Lorentz transformation in the ordinary space with the speed of light c changed to the effective speed $\tilde{c}$ [43].

Note that if relation (208) holds, the effective speed does not depend on mass

$$\frac{1}{\tilde{c}^2} = \frac{1}{c^2} - \frac{8}{3}\gamma^2 \tag{307}$$

and therefore, the Lorentz transformation does not depend on mass, either.

It is straightforward to generalize the obtained result to the case of the three-dimensional space (27), (28). In the frame of the algebra (27), (28) the Hamiltonian has the form (306) where the effective velocity is defined as

$$\frac{1}{\tilde{c}^2} = \frac{1}{c^2} - 4\gamma^2 \tag{308}$$

Therefore, the Lorentz transformation in the deformed space depends on the effective speed of light given by (308).

It is worth noting that a similar result can be obtained in the frame of different deformed algebras. The difference in the cases of different algebras is in the factor before γ^2. Namely, in general one can write

$$\frac{1}{\tilde{c}^2} = \frac{1}{c^2} - \frac{1}{u^2} \tag{309}$$

where $u = \alpha/\gamma$, α is a multiplier which is different for different algebras [43].

Taking into account (238) we can estimate that the relative deviation of the effective speed of light $\tilde{c}$ from c is very small

$$\frac{\tilde{c} - c}{c} = 2c^2\gamma^2 \simeq 3.5 \times 10^{-45} \tag{310}$$

At the end of this section we would like to note about the interpretation of $\tilde{c}$ and discuss the possibility to measure the discrepancy between $\tilde{c}$ and c. The additional constant $m(c^2-\tilde{c}^2)$ in the Hamiltonian (306) does not affect the motion. Therefore, the equations of motion of a relativistic particle in the deformed space and the Lorentz transformation depend on the effective speed of light $\tilde{c}$. For a massless particle we have $H=\tilde{c}p$. Therefore, the measured speed of light is the effective speed $\tilde{c}$. The speed of light c is related to the rest mass energy and can be treated as a bare speed of light. If $p=0$ from (306) we have $H=mc^2$. Therefore, the speed c is important in processes of annihilation of particles. Hence, a detailed analysis of the annihilation of the electron and the positron in the frame of deformed commutation relations gives the possibility to set an upper bound for the discrepancy between $\tilde{c}$ and c and estimate the minimal length. To obtain these results, experiments with high accuracy have to be performed [43].

3.6. Minimal length estimation based on studies of Mercury perihelion shift

The problem of the extremely small result for the minimal length obtained on the basis of the studies of the Mercury perihelion shift (see [29, 97]) disappear, if we take into account features of the description of motion of a macroscopic body in deformed space with a minimal length [45, 111].

Let us consider in details the perihelion shift of the Mercury planet in the frame of the algebra (31), (32) [96]. In the classical limit $\hbar\to 0$ from (31), (32) we have the following Poisson brackets

$$\{X_i,X_j\}=\{P_i,P_j\}=0 \tag{311}$$

$$\{X_i,P_j\}=\delta_{ij}(1+\beta P^2)+2\beta P_iP_j \tag{312}$$

Up to the first order in β the coordinates and momenta which satisfy (311), (312) can be represented as

$$X_i=x_i \tag{313}$$

$$P_i=p_i(1+\beta p^2) \tag{314}$$

where x_i, p_i satisfy the ordinary commutation relations (15), (16). Taking into account (313), (314) up to the first order in β the Hamiltonian of a particle with mass m in a gravitational field can be written as

$$H=\frac{P^2}{2m}-\frac{mk}{X}=\frac{p^2}{2m}-\frac{mk}{x}+\frac{\beta}{m}p^4 \tag{315}$$

where k is a constant, $X=\sqrt{\sum_i X_i^2}$, $x=\sqrt{\sum_i x_i^2}$. An additional term in the Hamiltonian $\beta p^4/m$ causes the perihelion shift of a particle. To find this shift it is convenient to consider the Hamilton vector defined as

$$\mathbf{u}=\frac{\mathbf{p}}{m}-\frac{mk[\mathbf{L}\times\mathbf{x}]}{xL^2} \tag{316}$$

(here $\mathbf{L}=[\mathbf{x}\times\mathbf{p}]$) and calculate its precession rate

$$\boldsymbol{\Omega}=\frac{[\mathbf{u}\times\dot{\mathbf{u}}]}{u^2} \tag{317}$$

Note that in the ordinary space ($\beta=0$) the Hamilton vector is preserved

$$\left\{\mathbf{u},\frac{p^2}{2m}-\frac{mk}{x}\right\}=0 \tag{318}$$

In the case of deformed space with the minimal length (31), (32), taking into account (315), the $\dot{\mathbf{u}}$ reads

$$\dot{\mathbf{u}}=\left\{\mathbf{u},\frac{\beta}{m}p^4\right\}=\frac{4\beta kp^2}{x^3}\mathbf{x} \tag{319}$$

Therefore, using (319) and taking into account that

$$u=\frac{mke}{L} \tag{320}$$

(here e is the eccentricity of the orbit) we obtain [96]

$$\Omega=\frac{4\beta p^2 L}{mx^3e^2}\left(x-\frac{L^2}{m^2k}\right) \tag{321}$$

In the ordinary space ($\beta=0$) we have

$$\begin{aligned}
&L=mx^2\dot{\phi}\\
&x=\frac{a(1-e^2)}{1+e\cos\phi}\\
&\frac{p^2}{2m}-\frac{mk}{x}=-\frac{mk}{2a}
\end{aligned} \tag{322}$$

where a is the semi-major axis, ϕ is the polar angle. Therefore, up to the first order in the parameter β, the perihelion shift per revolution reads [96]

$$\Delta\phi_p=\int_0^T\Omega dt=\int_0^{2\pi}\frac{\Omega}{\dot{\phi}}d\phi=-\frac{8\pi\beta m^2k}{a(1-e^2)} \tag{323}$$

We would like to mention that by reason of the assumption that the parameter of deformation is the same for elementary particles and macroscopic bodies the authors of the papers [29, 97] have obtained extremely small results for the minimal length in the deformed space on the basis of studies of the perihelion shift of the Mercury planet. For instance in [29] it was found that the upper bound on the minimal length in the space (10)–(12) is of the order 10^{-68}m. This result is well below the Planck length.

We would like to stress that if we consider the parameter of deformation corresponding to the Mercury planet to be the same as for the elementary particle, assuming that the minimal length is of the order of the Planck length $\hbar\sqrt{\beta}=10^{-35}$m and taking into account (323) we find

$$\Delta\phi_p = -\frac{8\pi\beta G M^2 M_S}{a(1-e^2)} = 2\pi\cdot 10^{55} \text{ radians/revolution} \tag{324}$$

where G is the gravitational constant, M_S is the mass of the Sun, M is the mass of the Mercury. It follows from (324) that the minimal length has a great effect on the motion of the Mercury planet which is not consistent with the observations.

The problem is solved if we take into account that the motion of the center-of-mass of a body in the deformed space is described by the effective parameter (201). Therefore, the perihelion shift of a macroscopic body can be found replacing β in (323) to $\tilde{\beta}$ (201) [45, 111]. For the Mercury planet we have

$$\Delta\phi_p = -\frac{8\pi\tilde{\beta} G M^2 M_S}{a(1-e^2)} \tag{325}$$

The perihelion precession rate which cannot be explained by the Newtonian gravitational effects of other planets and asteroids, Solar Oblateness and which is usually explained by relativistic effects (Lense-Thirring, gravitoelectric effect) is

$$\begin{aligned} &\Delta\phi_{obs} = 42.9779 \pm 0.0009 \text{ arc}-\text{seconds per century} = \\ &2\pi(7.98695 \pm 0.00017)\cdot 10^{-8} \text{ radians/revolution} \end{aligned} \tag{326}$$

(see table 3 in [112]). From the General Relativity predictions the perihelion precession rate is

$$\Delta\phi_{GR} = 2\pi(7.98744\cdot 10^{-8}) \text{ radians/revolution} \tag{327}$$

(see, for instance, [29]).

Comparing the perihelion shift caused by noncommutativity with

$$\Delta\phi_{obs} - \Delta\phi_{GR} = 2\pi(-0.00049 \pm 0.00017)\cdot 10^{-8} \text{ radians/revolution} \tag{328}$$

and assuming that

$$|\Delta\phi_{nc}| \le |\Delta\phi_{obs} - \Delta\phi_{GR}| \tag{329}$$

at 3σ one can write

$$|\Delta\phi_p| \le 2\pi\cdot 10^{-11} \text{ radians/revolution} \tag{330}$$

Taking into account (257), the effective parameter $\tilde{\beta}$ is related to the parameter of deformation corresponding to a particle as

$$\tilde{\beta} = \frac{\beta_p m_p^2}{M^2} \tag{331}$$

where m_p is the mass of the particle. Hence, using (331), on the basis of inequality (330) for the parameter of deformation corresponding to nucleons we find [45, 111]

$$\hbar\sqrt{\beta_{nuc}} < 2\cdot 10^{-18} m \tag{332}$$

Similarly, for the minimal length corresponding to the electron, we obtain

$$\hbar\sqrt{\beta_{e}} < 3.7\cdot 10^{-15} m \tag{333}$$

This result is not so strong as that obtained on the basis of studies of the hydrogen atom in the deformed space in [45, 113, 114]. This is because the effect of minimal length on the motion of macroscopic bodies is less than this effect on the elementary particles (202). Therefore, to find a strong restriction on the minimal length on the basis of studies of macroscopic bodies the results with a high precision are needed.

At the end of this section we would like to note that the expression for the perihelion shift (323) depends on the mass of a particle (expression (325) depends on the mass of the Mercury). This is a consequence of violation of the weak equivalence principle in deformed space. It is important to stress that if the condition (257) holds, the expression for the perihelion shift does not depend on the mass. We have

$$\Delta\phi_p = -\frac{8\pi\gamma^2 k}{a(1-e^2)} \tag{334}$$

and the weak equivalence principle is preserved.

Hence, the relation of the parameter of deformation and mass opens a possibility to solve the list of problems in the deformed space with a minimal length. These problems include the soccer-ball problem, violation of the properties of the kinetic energy, violation of the weak equivalence principle, the dependence of the Galilean and Lorentz transformations on mass.

In the next chapters we show that the idea to relate parameters of deformed algebras to mass is also important in the noncommutative space of a canonical type, in spaces with the Lie algebraic noncommutativity, in the twist-deformed space.

4. Many-particle system in noncommutative phase space of canonical type

In this chapter we present features of the description of the motion of a composite system in a noncommutative phase space of a canonical type in a general case when the coordinates and the momenta of different particles satisfy the noncommutative algebra with different parameters. We show that the motion of a composite system in a noncommutative phase space is described by effective parameters of noncommutativity, the motion of the center-of-mass of a composite system is not independent of its relative motion, free particles do not move together in the noncommutative phase space, even if the initial velocities of the particles are the same, the properties of kinetic energy (property of additivity, independence of composition) are violated because of the noncommutativity. We conclude that all these problems can be solved if we assume that the parameters of coordinate noncommutativity are inversely proportional to mass and the parameters of momentum noncommutativity are proportional to mass. Besides, due to these relations of parameters of noncommutativity with mass, the weak equivalence principle is preserved in the noncommutative phase space.

The chapter is organized as follows. In Section 4.1 we analyze the commutation relations for the coordinates and momenta of the center-of-mass and the relative motion in the four-dimensional noncommutative phase space (2D configurational and 2D momentum space) of a canonical type. In Section 4.2 the conditions on the parameters of noncommutativity which give a possibility to consider the motion of the center-of-mass independently of the relative motion are found. Section 4.3 is devoted to studies of the free particles system motion in a noncommutative phase space of a canonical type. In Section 4.4 the weak equivalence principle is studied in a noncommutative phase space. In Section 4.5 the properties of the kinetic energy of a composite system are discussed in the frame of a noncommutative algebra of a canonical type. The total momentum as the integral of motion in the noncommutative phase space is introduced in Section 4.6. Section 4.7 is devoted to a generalization of the obtained results to the case of six-dimensional (3D configurational and 3D momentum space) noncommutative phase space of a canonical type. In Section 4.8 the upper bounds for the parameters of noncommutativity are found on the basis of studies of the perihelion shift of the Mercury planet. In Section 4.9 the influence of noncommutativity on the motion of the Earth and the Moon is considered and the weak equivalence principle is examined.

4.1. Noncommutative algebra for coordinates and momenta of the center-of-mass and relative motion

Let us consider the noncommutative algebra of a canonical type (73)–(75) with $\sigma = 0$ and write this algebra for coordinates and momenta of different particles labeled by indexes a and b. In a general case, the parameters of the algebra (73)–(75) corresponding to different particles can be different. Hence, we can write

$$[X_1^{(a)}, X_2^{(b)}] = i\hbar\delta^{ab}\theta^{(a)} \tag{335}$$

$$[X_i^{(a)}, P_j^{(b)}] = i\hbar\delta^{ab}\delta_{ij} \tag{336}$$

$$[P_1^{(a)}, P_2^{(b)}] = i\hbar\delta^{ab}\eta^{(a)} \tag{337}$$

where $i = (1,2)$, $j = (1,2)$, the parameters $\theta^{(a)}$, $\eta^{(a)}$, $\gamma^{(a)}$ correspond to a particle labeled by index a. Writing (335)–(337) we assume that the coordinates and the momenta of different particles commute. In the classical limit $\hbar \to 0$ from (335)–(337) one obtains the following Poisson brackets

$$\{X_1^{(a)}, X_2^{(b)}\} = \delta^{ab}\theta_a \tag{338}$$

$$\{X_i^{(a)}, P_j^{(b)}\} = \delta^{ab} \tag{339}$$

$$\{P_1^{(a)}, P_2^{(b)}\} = \delta^{ab}\eta_a \tag{340}$$

For the coordinates and momenta of the center-of-mass introduced in the traditional way (259), (260), taking into account (338)–(340), we can write

$$\{\tilde{X}_1, \tilde{X}_2\} = \tilde{\theta} \tag{341}$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \delta^{ab}\delta_{ij} \tag{342}$$

$$\{\tilde{P}_1, \tilde{P}_2\} = \tilde{\eta} \tag{343}$$

Here, we use the notations $\tilde{\theta}$, $\tilde{\eta}$, for effective parameters which describe the motion of the center-of-mass of a composite system and are defined as [115, 116]

$$\tilde{\theta} = \frac{\sum_a m_a^2 \theta_a}{(\sum_b m_b)^2} \tag{344}$$

$$\tilde{\eta} = \sum_a \eta_a \tag{345}$$

It is worth noting that the effective parameters (344), (345) depend on the masses of particles which form the system and on the parameters of noncommutativity θ_a, η_a, corresponding to the individual particles. Therefore, $\tilde{\theta}$, $\tilde{\eta}$, depend on the system's composition. It is important to mention that there is a reduction in the effective parameter of coordinate noncommutativity with respect to the parameters of noncommutativity corresponding to individual particles. For a system of N particles with the same masses m and parameters of noncommutativity θ we have

$$\tilde{\theta} = \frac{\theta}{N} \tag{346}$$

The effective parameter of momentum noncommutativity increases with the increasing number of particles in a system. For a system of N particles with the same masses m this parameter reads

$$\tilde{\eta} = N\eta \tag{347}$$

Let us also introduce the coordinates and momenta of the relative motion in the traditional way

$$\Delta \mathbf{X}^{(a)} = \mathbf{X}^{(a)} - \tilde{\mathbf{X}} \tag{348}$$

$$\Delta \mathbf{P}^{a} = \mathbf{P}^{(a)} - \mu_a \tilde{\mathbf{P}} \tag{349}$$

They satisfy the following relations

$$\{\Delta X_1^{(a)},\Delta X_2^{(b)}\}=-\{\Delta X_2^{(a)},\Delta X_1^{(b)}\}=\delta^{ab}\theta_a-\mu_a\theta_a-\mu_b\theta_b+\tilde{\theta} \tag{350}$$

$$\{\Delta P_1^{(a)},\Delta P_2^{(b)}\}=-\{\Delta P_2^{(a)},\Delta P_1^{(b)}\}=\delta^{ab}\eta_a-\mu_b\eta_a-\mu_a\eta_b+\mu_a\mu_b\tilde{\eta} \tag{351}$$

$$\{\Delta X_i^{(a)},\Delta P_j^{(b)}\}=\delta_{ij}(\delta_{ab}-\mu_b) \tag{352}$$

It is important that for the coordinates and momenta of the center-of-mass and the coordinates and momenta of the relative motion we have

$$\{\tilde{X}_1,\Delta X_2^{(a)}\}=-\{\tilde{X}_2,\Delta X_1^{(a)}\}=\mu_a\theta_a-\tilde{\theta} \tag{353}$$

$$\{\tilde{P}_1,\Delta P_2^{(a)}\}=-\{\tilde{P}_2,\Delta P_1^{a}\}=\eta_a-\mu_a\sum_b\eta_b \tag{354}$$

It follows from (353), (354) that we cannot study the motion of the center-of-mass of a composite system independently of the relative motion. The relative motion is influenced by the motion of the center-of-mass and vice versa. Therefore, the two-particle problem cannot be reduced to the one-particle problem [115, 116].

In the next section we show that the motion of the center-of-mass is independent of the relative motion and the two-particle problem can be reduced to the one-particle problem, if we consider the parameters of noncommutativity to be dependent on mass.

4.2. Reduction of two-particle problem to one-particle problem in noncommutative phase space

Let us consider a system of two particles of masses m_1, m_2 which is described by the following Hamiltonian

$$H=\frac{(\mathbf{P}^{(1)})^2}{2m_1}+\frac{(\mathbf{P}^{(2)})^2}{2m_2}+U(|\mathbf{X}^{(1)}-\mathbf{X}^{(2)}|) \tag{355}$$

Here $U(|\mathbf{X}^{(1)}-\mathbf{X}^{(2)}|)$ is the interaction potential energy. The coordinates and momenta of particles satisfy the relations (338)–(340). The Hamiltonian (355) can be rewritten as follows

$$H=\frac{(\tilde{\mathbf{P}})^2}{2M}+\frac{(\mathbf{P}^r)^2}{2\mu}+U(|\mathbf{X}^r|) \tag{356}$$

where $M=m_1+m_2$ is the total mass and $\mu=m_1m_2/(m_1+m_2)$ is the reduced mass, $\tilde{P}_i$ are the momenta of the center-of-mass defined in the traditional way (260) and

$$\begin{aligned}\mathbf{P}^r&=\frac{1}{2}\left(\Delta\mathbf{P}^{(2)}-\Delta\mathbf{P}^{(1)}\right)=\mu_1\mathbf{P}^{(2)}-\mu_2\mathbf{P}^{(1)}\\ \mathbf{X}^r&=\Delta\mathbf{X}^{(2)}-\Delta\mathbf{X}^{(1)}=\mathbf{X}^{(2)}-\mathbf{X}^{(1)}\end{aligned} \tag{357}$$

Due to the relations

$$\{\tilde{P}_1,P_2^r\}=-\{\tilde{P}_2,P_1^r\}=\mu_1\eta_2-\mu_2\eta_1 \tag{358}$$

$$\{\tilde{X}_1, X_2^r\} = -\{\tilde{X}_2, X_1^r\} = \mu_2\theta_2 - \mu_1\theta_1 \tag{359}$$

the two-particle problem cannot be reduced to the one-particle problem for the internal motion. The terms $(\tilde{\mathbf{P}})^2/2M$, $(\mathbf{P}^r)^2/2\mu + U(|\mathbf{X}^r|)$ in the Hamiltonian (356) cannot be considered separately.

It is important to stress that the Poisson brackets (358), (359) (also (353), (354)) are equal to zero if the following relations are satisfied

$$\theta_a m_a = \gamma = const \tag{360}$$

$$\frac{\eta_a}{m_a} = \alpha = const \tag{361}$$

where γ, α are constants which are the same for different particles [115].[3]

Thus, if the parameters of noncommutativity which correspond to a particle are determined by its mass as (360), (361) the motion of the center-of-mass of a composite system is independent of its relative motion and the two-particle problem can be reduced to the one-particle problem. It is also worth mentioning that in the case when the conditions (360), (361) are satisfied the effective parameters $\tilde{\theta}$, $\tilde{\eta}$ depend only on the total mass of the system and do not depend on its composition. From (344), (345), taking into account (360), (361), we have

$$m_a\theta_a = \tilde{\theta}M = \gamma \tag{362}$$

$$\frac{\eta_a}{m_a} = \frac{\tilde{\eta}}{M} = \alpha \tag{363}$$

Thus, the conditions (360), (361) are also satisfied for the effective parameters of noncommutativity. If the parameters of the coordinate noncommutativity corresponding to individual particles are assumed to be inversely proportional to their masses and the parameters of momentum noncommutativity of individual particles are assumed to be proportional to their masses, one obtains the same dependence of the effective parameters of noncommutativity corresponding to a composite system on the system's total mass (362), (363) [115].

In the next section we show that the relations (360), (361) are important for recovering the independence of the motion of the free particle of mass.

4.3. Influence of noncommutativity on free particles system motion in four-dimensional noncommutative phase space

For a free particle with mass m, considering the Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} \tag{364}$$

and taking into account (338)–(340) we obtain the following equations of motion

$$\dot{X}_1 = \frac{P_1}{m}, \quad \dot{X}_2 = \frac{P_2}{m} \tag{365}$$

3. Note that in (360) the constant γ is not the same as in the relations (208), (257).

$$\dot{P}_1 = \eta\frac{P_2}{m}, \quad \dot{P}_2 = -\eta\frac{P_1}{m} \tag{366}$$

The solutions of these equations are

$$X_1(t) = \upsilon_{01}\frac{m}{\eta}\sin\frac{\eta}{m}t - \upsilon_{02}\frac{m}{\eta}\cos\frac{\eta}{m}t + \upsilon_{02}\frac{m}{\eta} + X_{01} \tag{367}$$

$$X_2(t) = \upsilon_{02}\frac{m}{\eta}\sin\frac{\eta}{m}t + \upsilon_{01}\frac{m}{\eta}\cos\frac{\eta}{m}t - \upsilon_{01}\frac{m}{\eta} + X_{02} \tag{368}$$

$$\dot{X}_1(t) = \upsilon_{01}\cos\frac{\eta}{m}t + \upsilon_{02}\sin\frac{\eta}{m}t \tag{369}$$

$$\dot{X}_2(t) = \upsilon_{02}\cos\frac{\eta}{m}t - \upsilon_{01}\sin\frac{\eta}{m}t \tag{370}$$

Here, we consider the initial conditions

$$X_1(0) = X_{01}, \quad X_2(0) = X_{02} \tag{371}$$

$$\dot{X}_1(0) = \upsilon_{01}, \quad \dot{X}_2(0) = \upsilon_{02} \tag{372}$$

It is important to mention that the noncommutativity of momenta causes the dependence of the trajectory and velocity of a free particle (367)–(370) on its mass. In the limit $\theta \to 0$, $\eta \to 0$ from (367)–(370) we obtain the well known expressions for the trajectory of a free particle in the ordinary space

$$X_1(t) = \upsilon_{01}t + X_{01}, \quad X_2(t) = \upsilon_{02}t + X_{02} \tag{373}$$

In the case of a system of N free particles with masses m_1, m_2,...,m_N the Hamiltonian reads

$$H = \sum_a \frac{(\mathbf{P}^{(a)})^2}{2m_a} = \frac{\tilde{\mathbf{P}}^2}{2M} + \sum_a \frac{(\Delta\mathbf{P}^{(a)})^2}{2m_a} \tag{374}$$

Here, the momenta $P_i^{(a)}$ satisfy the relation (340), the index a labels the particles, M is the total mass of the system $M = \sum_a m_a$. The momenta $\tilde{P}_i$, $\Delta P_i^{(a)}$ corresponding to the center-of-mass and relative motion satisfy (343), (351), (354).

In the ordinary space ($\theta = 0$, $\eta = 0$), if the initial velocities of free particles are the same, free particles move together. In the noncommutative phase space, even if the initial velocities of free particles are the same

$$\dot{X}_1^{(a)}(0) = \upsilon_{01}, \quad \dot{X}_2^{(a)}(0) = \upsilon_{02} \tag{375}$$

$a = (1...N)$, using (369), (370), we have

$$\dot{X}_1^{(a)}(t) = \upsilon_{01}\cos\frac{\eta_a}{m_a}t + \upsilon_{02}\sin\frac{\eta_a}{m_a}t \tag{376}$$

$$\dot{X}_2^{(a)}(t) = \upsilon_{02}\cos\frac{\eta_a}{m_a}t - \upsilon_{01}\sin\frac{\eta_a}{m_a}t \tag{377}$$

where η_a is the parameter of momentum noncommutativity which corresponds to the particle with mass m_a, $a = (1..N)$. Note that $\dot{X}_1^{(a)}(t) \neq \dot{X}_1^{(b)}(t)$, $\dot{X}_2^{(a)}(t) \neq$

$\dot{X}_2^{(b)}(t)$ for $a \neq b$. Thus, due to the noncommutativity of the momenta the system of free particles with the same initial velocities flies away [117].

It is important to stress that due to the relation (354), the relative motion affects the motion of the center-of-mass even in the case of a system of free particles. The trajectories corresponding to the motion of the center-of-mass and to the relative motion read

$$\tilde{X}_1(t) = \sum_a \left(\upsilon_{01} \frac{m_a^2}{M\eta_a} \sin\frac{\eta_a}{m_a} t - \upsilon_{02} \frac{m_a^2}{M\eta_a} \cos\frac{\eta_a}{m_a} t + \upsilon_{02} \frac{m_a^2}{M\eta_a} + \frac{m_a}{M} X_{01}^{(a)} \right) \tag{378}$$

$$\tilde{X}_2(t) = \sum_a \left(\upsilon_{02} \frac{m_a^2}{M\eta_a} \sin\frac{\eta_a}{m_a} t + \upsilon_{01} \frac{m_a^2}{M\eta_a} \cos\frac{\eta_a}{m_a} t - \upsilon_{01} \frac{m_a^2}{M\eta_a} + \frac{m_a}{M} X_{02}^{(a)} \right) \tag{379}$$

$$\Delta X_1^a(t) = \upsilon_{01} \frac{m_a}{\eta_a} \sin\frac{\eta_a}{m_a} t - \upsilon_{02} \frac{m_a}{\eta_a} \cos\frac{\eta_a}{m_a} t + \upsilon_{02} \frac{m_a}{\eta_a} + X_{01}^{(a)} -$$
$$\sum_b \left(\upsilon_{01} \frac{m_b^2}{M\eta_b} \sin\frac{\eta_b}{m_b} t - \upsilon_{02} \frac{m_b^2}{M\eta_b} \cos\frac{\eta_b}{m_b} t + \upsilon_{02} \frac{m_b^2}{M\eta_b} + \frac{m_b}{M} X_{01}^{(b)} \right) \tag{380}$$

$$\Delta X_2^a(t) = \upsilon_{02} \frac{m_a}{\eta_a} \sin\frac{\eta_a}{m_a} t + \upsilon_{01} \frac{m_a}{\eta_a} \cos\frac{\eta_a}{m_a} t - \upsilon_{01} \frac{m_a}{\eta_a} + X_{02}^{(a)} -$$
$$\sum_b \left(\upsilon_{02} \frac{m_b^2}{M\eta_b} \sin\frac{\eta_b}{m_b} t + \upsilon_{01} \frac{m_b^2}{M\eta_b} \cos\frac{\eta_b}{m_b} t - \upsilon_{01} \frac{m_b^2}{M\eta_b} + \frac{m_b}{M} X_{02}^{(b)} \right) \tag{381}$$

From (376), (377), (380), (381) we have that the free particles do not move together.

Note that if the parameter of the momentum noncommutativity is related to mass as (361), the trajectories of free particles do not depend on their masses. From (367), (368), we obtain [117]

$$X_1^{(a)}(t) = \frac{\upsilon_{01}}{\alpha} \sin\alpha t - \frac{\upsilon_{02}}{\alpha} \cos\alpha t + \frac{\upsilon_{02}}{\alpha} + X_{01}^{(a)} \tag{382}$$

$$X_2^{(a)}(t) = \frac{\upsilon_{02}}{\alpha} \sin\alpha t + \frac{\upsilon_{01}}{\alpha} \cos\alpha t - \frac{\upsilon_{01}}{\alpha} + X_{02}^{(a)} \tag{383}$$

where

$$X_{01}^{(a)} = X_1^{(a)}(0), \quad X_{02}^{(a)} = X_2^{(a)}(0) \tag{384}$$

Also, if the condition (361) holds the particles forming the system and the center-of-mass of the system move with the same velocities

$$\begin{aligned} \dot{X}_1^{(a)}(t) &= \sum_a \mu_a \dot{X}_1^{(a)}(t) = \upsilon_{01} \cos\alpha t + \upsilon_{02} \sin\alpha t \\ \dot{X}_2^{(a)}(t) &= \sum_a \mu_a \dot{X}_2^{(a)}(t) = \upsilon_{02} \cos\alpha t - \upsilon_{01} \sin\alpha t \end{aligned} \tag{385}$$

and from (380), (381) we have

$$\Delta X_1^{(a)} = X_{01}^{(a)} - \sum_b \mu_b X_{01}^{(b)}$$
$$\Delta X_2^{(a)} = X_{02}^{(a)} - \sum_b \mu_b X_{02}^{(b)} \tag{386}$$

The relative coordinates are constants and particles move together as it is in the ordinary space ($\theta = \beta = 0$) [117].

In the next section we show that due to the condition (360), (361) the weak equivalence principle is preserved in the noncommutative phase space.

4.4. Weak equivalence principle in four-dimensional noncommutative phase space

Studies of the influence of noncommutativity on the implementation of the weak equivalence principle are presented in [118, 119, 67, 120, 121, 116, 115, 122]. The authors of the paper [121] conclude that the equivalence principle holds in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, unless the noncommutative parameters are anisotropic. In this section we show that the weak equivalence principle can be recovered in the noncommutative phase space of a canonical type due to the relations (360), (361).

Let us first study the motion of a particle with mass m in a uniform gravitational field directed along the X_1 axis in the frame of the noncommutative algebra (338)–(340). Considering the Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - mgX_1 \tag{387}$$

and taking into account (338)–(340) we can write the following equations of motion

$$\dot{X}_1 = \frac{P_1}{m} \tag{388}$$

$$\dot{X}_2 = \frac{P_2}{m} + mg\theta \tag{389}$$

$$\dot{P}_1 = mg + \eta\frac{P_2}{m} \tag{390}$$

$$\dot{P}_2 = -\eta\frac{P_1}{m} \tag{391}$$

Considering the initial conditions (371), (372), from (388)–(391) we have

$$X_1(t) = \frac{m\upsilon_{01}}{\eta}\sin\frac{\eta}{m}t + \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\left(1 - \cos\frac{\eta}{m}t\right) + X_{01} \tag{392}$$

$$X_2(t) = \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\sin\frac{\eta}{m}t - \frac{m\upsilon_{01}}{\eta}\left(1 - \cos\frac{\eta}{m}t\right) -$$

$$\frac{mg}{\eta}t+mg\theta t+X_{02} \tag{393}$$

It follows from (392), (393) that if we consider the parameters of noncommutativity to be the same for different particles, the motion of a particle in the uniform gravitational field depends on its mass. Therefore, the weak equivalence principle is violated in the noncommutative phase space.

Note that the relations (360), (361) lead to recovering the weak equivalence principle in the noncommutative phase space [115, 116]. If the relations (360), (361) are satisfied, the trajectory of a particle in a gravitational field reads

$$X_1(t)=\frac{\upsilon_{01}}{\alpha}\sin\alpha t+\left(\frac{g}{\alpha^2}-\frac{g\gamma}{\alpha}+\frac{\upsilon_{02}}{\alpha}\right)(1-\cos\alpha t)+X_{01} \tag{394}$$

$$X_2(t)=\left(\frac{g}{\alpha^2}-\frac{g\gamma}{\alpha}+\frac{\upsilon_{02}}{\alpha}\right)\sin\alpha t-\frac{\upsilon_{01}}{\alpha}(1-\cos\alpha t)-\frac{g}{\alpha}t+\gamma gt+X_{02} \tag{395}$$

The trajectory depends on the constants α, γ and is the same for particles with different masses.

In the case of the motion of a particle in the nonuniform field $V(X_1,X_2)$

$$H=\frac{P_1^2}{2m}+\frac{P_2^2}{2m}+mV(X_1,X_2) \tag{396}$$

taking into account (338)–(340), we obtain the following equations of motion

$$\dot{X}_1=\frac{P_1}{m}+m\theta\frac{\partial V(X_1,X_2)}{\partial X_2} \tag{397}$$

$$\dot{X}_2=\frac{P_2}{m}-m\theta\frac{\partial V(X_1,X_2)}{\partial X_1} \tag{398}$$

$$\dot{P}_1=-m\frac{\partial V(X_1,X_2)}{\partial X_1}+\eta\frac{P_2}{m} \tag{399}$$

$$\dot{P}_2=-m\frac{\partial V(X_1,X_2)}{\partial X_2}-\eta\frac{P_1}{m} \tag{400}$$

If the conditions (360), (361) are satisfied we can write

$$\dot{X}_1=P_1'+\gamma\frac{\partial V(X_1,X_2)}{\partial X_2} \tag{401}$$

$$\dot{X}_2=P_2'-\gamma\frac{\partial V(X_1,X_2)}{\partial X_1} \tag{402}$$

$$\dot{P}_1'=-\frac{\partial V(X_1,X_2)}{\partial X_1}+\alpha P_2' \tag{403}$$

$$\dot{P}_2'=-\frac{\partial V(X_1,X_2)}{\partial X_2}-\alpha P_1' \tag{404}$$

with

$$P_i' = \frac{P_i}{m} \tag{405}$$

Note that the equations (401)–(404) do not contain mass, therefore, solutions of these equations $X_i = X_i(t)$, $P_i' = P_i'(t)$ do not depend on mass. Hence, we can conclude that due to the relations (360), (361) the kinematic characteristics of the particle do not depend on its mass and the weak equivalence principle is recovered.

In the case of the motion of a composite system (body) with mass M in the gravitational field $V(\tilde{X}_1, \tilde{X}_2)$ we have the following Hamiltonian

$$H = \frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2) + H_{rel} \tag{406}$$

where $\tilde{X}_i$, $\tilde{P}_i$, are the coordinates and momenta of the center-of-mass (259), (260) which satisfy the relations (341)–(343) with effective parameters of noncommutativity. It is worth mentioning at this point that according to the definition (344), (345) these parameters depend on the composition of a system (body). Thus, the effect of noncommutativity on composite systems of the same masses but different compositions is different. This is an additional cause of violation of the weak equivalence principle in a space with noncommutativity of coordinates and noncommutativity of momenta. The term H_{rel} in (406) corresponds to the relative motion and depends on the coordinates and momenta of the relative motion.

If the conditions (360), (361) are satisfied we have that the effective parameters of noncommutativity do not depend on the composition (see (362), (363)) also

$$\left\{ \frac{\tilde{\mathbf{P}}^2}{2M} + MV(\tilde{X}_1, \tilde{X}_2), H_{rel} \right\} = 0 \tag{407}$$

Therefore, we obtain the equations of the motion of the center-of-mass as follows

$$\begin{aligned}
\dot{\tilde{X}}_1 &= \tilde{P}_1' + \gamma \frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2} \\
\dot{\tilde{X}}_2 &= \tilde{P}_2' - \gamma \frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1} \\
\dot{\tilde{P}}_1' &= -\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_1} + \alpha \tilde{P}_2' \\
\dot{\tilde{P}}_2' &= -\frac{\partial V(\tilde{X}_1, \tilde{X}_2)}{\partial \tilde{X}_2} - \alpha \tilde{P}_1'
\end{aligned} \tag{408}$$

where $\tilde{P}_i' = \tilde{P}_i'/M$.

Thus, we can conclude that due to the relation of the parameters of noncommutativity with mass (360), (361) the motion of a body in a gravitational field does not depend on its mass and composition, and the weak equivalence principle is preserved [115, 116].

4.5. Properties of kinetic energy of composite system in noncommutative phase space

Let us consider a system of particles in the noncommutative phase space and study the case when each particle which forms the system moves with the same velocity as its center-of-mass. This case is equivalent to the case when a macroscopic body can be divided into N parts which can be considered as particles. The kinetic energy of the body of mass M is the following

$$T=\frac{\tilde{P}_1^2}{2M}+\frac{\tilde{P}_2^2}{2M} \tag{409}$$

where the momenta of the center-of-mass satisfy the relations of the noncommutative algebra (343). Taking into account the fact that for a body in the uniform gravitational field the momenta are given by

$$\tilde{P}_1=M\tilde{\upsilon}_{01}\cos\frac{\tilde{\eta}}{M}t+(M\tilde{\upsilon}_{02}+\frac{M^2g}{\tilde{\eta}}-M^2g\tilde{\theta})\sin\frac{\tilde{\eta}}{M}t \tag{410}$$

$$\tilde{P}_2=-M\tilde{\upsilon}_{01}\sin\frac{\tilde{\eta}}{M}t+(M\tilde{\upsilon}_{02}+\frac{M^2g}{\tilde{\eta}}-M^2g\tilde{\theta})\cos\frac{\tilde{\eta}}{M}t-\frac{M^2g}{\tilde{\eta}} \tag{411}$$

(where $\tilde{\upsilon}_{01}$, $\tilde{\upsilon}_{02}$ are initial velocities of the center-of-mass of the body, we use (388)–(391) and (392), (393)), the kinetic energy can be rewritten as

$$T=T_0+g^2M^3\left(\frac{1}{\tilde{\eta}^2}+\frac{\tilde{\theta}^2}{2}-\frac{\tilde{\theta}}{\tilde{\eta}}\right)+M^2g\tilde{\upsilon}_{02}\left(\frac{1}{\tilde{\eta}}-\tilde{\theta}\right)+$$
$$\frac{M^2g}{\tilde{\eta}}\left(\tilde{\upsilon}_{01}\sin\frac{\tilde{\eta}}{M}t+\left(\frac{Mg}{\tilde{\eta}}-Mg\tilde{\theta}+\tilde{\upsilon}_{02}\right)\cos\frac{\tilde{\eta}}{M}t\right) \tag{412}$$

where

$$T_0=\frac{M(\tilde{\upsilon}_{01}^2+\tilde{\upsilon}_{02}^2)}{2} \tag{413}$$

is the kinetic energy of the system in the ordinary space.

Note that the expression for the kinetic energy of the body (412) depends on the effective parameters of noncommutativity $\tilde{\theta}$, $\tilde{\eta}$ which depend on the composition of the body (344), (345). Thus, the property of independence of the kinetic energy of the composition is not satisfied. It is worth noting that the property of additivity of the kinetic energy is also violated in the noncommutative phase space. According to this property, for a body composed of N particles with masses m_a, $a=1..N$, we can write

$$T=\sum_a T_a=\sum_a\left[T_{0a}+g^2m_a^3\left(\frac{1}{\eta_a^2}+\frac{\theta_a^2}{2}-\frac{\theta_a}{\eta_a}\right)+m_a^2g\tilde{\upsilon}_{02}\left(\frac{1}{\eta_a}-\theta_a\right)+\right.$$
$$\left.\frac{m_a^2g}{\eta_a}\left(\tilde{\upsilon}_{01}\sin\frac{\eta_a}{m_a}t+\left(\frac{m_ag}{\eta_a}-m_ag\theta_a+\tilde{\upsilon}_{02}\right)\cos\frac{\eta_a}{m_a}t\right)\right] \tag{414}$$

Here, we take into account the fact that the particles, forming the body, move with the same velocities as the whole body.

It is important to stress that we obtain different expressions for the kinetic energy (412), (414). Note that if the relations (360), (361) hold, the expressions (412) and (414) are the same. We have

$$T=T_0+\sum_a m_a\left[g^2\left(\frac{1}{\alpha^2}+\frac{\gamma^2}{2}-\frac{\gamma}{\alpha}\right)+g\tilde{\upsilon}_{02}\left(\frac{1}{\alpha}-\gamma\right)+\right.$$
$$\left.\frac{g}{\alpha}\left(\tilde{\upsilon}_{01}\sin\alpha t+\left(\frac{g}{\alpha}-g\gamma+\tilde{\upsilon}_{02}\right)\cos\alpha t\right)\right] \tag{415}$$

Thus, the properties of the kinetic energy are preserved in the noncommutative phase space due to the relations (360), (361) [115, 116].

4.6. Total momentum as integral of motion in noncommutative phase space of canonical type

The momentum of the center-of-mass of a composite system defined as the sum of the momenta of the particles forming it (260) is not the integral of motion in the noncommutative phase space of a canonical type. Considering a composite system with the Hamiltonian

$$H=\sum_a\frac{(\mathbf{P}^{(a)})^2}{2m_a}+\frac{1}{2}\sum_{\substack{a,b\\a\neq b}}U(|\mathbf{X}^{(a)}-\mathbf{X}^{(b)}|) \tag{416}$$

for the momenta of the center-of-mass defined in the traditional way (260) we find

$$\begin{aligned}\{\tilde{P}_1,H\}&=\tilde{\eta}\frac{\tilde{P}_2}{M}+\sum_a\frac{\Delta P_2^{(a)}}{m_a}(\eta_a-\mu_a\tilde{\eta})\\ \{\tilde{P}_2,H\}&=-\tilde{\eta}\frac{\tilde{P}_1}{M}-\sum_a\frac{\Delta P_1^{(a)}}{m_a}(\eta_a-\mu_a\tilde{\eta})\end{aligned} \tag{417}$$

If the conditions (362), (363) are satisfied, these relations are reduced to

$$\begin{aligned}\{\tilde{P}_1,H\}&=\frac{\tilde{P}_2}{M}\tilde{\eta}\\ \{\tilde{P}_2,H\}&=-\frac{\tilde{P}_1}{M}\tilde{\eta}\end{aligned} \tag{418}$$

but do not vanish.

To find the integral of motion in the noncommutative phase space, let us first consider a particular case of a composite system made of N particles with masses $m_a = m$ and parameters $\theta_a = \theta$, $\eta_a = \eta$. Note that the following relation is satisfied

$$\left\{\sum_a P_1^{(a)} - \eta\sum_a X_2^{(a)}, H\right\} = \left\{\sum_a P_2^{(a)} + \eta\sum_a X_1^{(a)}, H\right\} = 0 \tag{419}$$

Hence, the values

$$\tilde{P}_1' = \sum_a P_1^{(a)} - \eta\sum_a X_2^{(a)} \tag{420}$$

$$\tilde{P}_2' = \sum_a P_2^{(a)} + \eta\sum_a X_1^{(a)} \tag{421}$$

are integrals of motion and can be considered as total momenta [117]. For $\eta \to 0$ the expressions (420), (421) transform to the total momenta defined in the traditional way (260).

In a general case, for a composite system made of N particles with different masses m_a, if the conditions (360), (361) hold, the integrals of motion are the following

$$\tilde{P}_1' = \tilde{P}_1 - \tilde{\eta}\tilde{X}_2 \tag{422}$$

$$\tilde{P}_2' = \tilde{P}_2 + \tilde{\eta}\tilde{X}_1 \tag{423}$$

where $\tilde{P}_i$, $\tilde{X}_i$ are the momenta and coordinates of the center-of-mass defined in the traditional way (259), (260), and $\tilde{\eta}$ is given by (345) [117]. Note that if the masses of the particles are the same $\tilde{\eta} = N\eta$, and the expressions (422), (423) transform to (420), (421).

The coordinates defined as

$$\tilde{X}_i' = \frac{\sum_a \mu_a X_i^{(a)}}{1 - \tilde{\eta}\tilde{\theta}} = \frac{\tilde{X}_i}{1 - \tilde{\eta}\tilde{\theta}} \tag{424}$$

are conjugated to $\tilde{P}_i'$, namely

$$\{\tilde{X}_i', \tilde{P}_j'\} = \delta_{ij} \tag{425}$$

Thus, the coordinates $\tilde{X}_i'$ can be treated as the coordinates of the center-of-mass [117]. For $\tilde{X}_i'$, $\tilde{P}_i'$ we also have

$$\{\tilde{X}_1', \tilde{X}_2'\} = \frac{\tilde{\theta}}{(1 - \tilde{\theta}\tilde{\eta})^2} \tag{426}$$

$$\{\tilde{P}_1', \tilde{P}_2'\} = \tilde{\eta}(\tilde{\theta}\tilde{\eta} - 1) \tag{427}$$

It is worth noting that even for a free particle in the noncommutative phase space the momentum is not an integral of motion. Taking into account (422), (423), (425) for a one-particle system we have the following integrals of motion $P_1' = P_1 - \eta X_2$, $P_2' = P_2 + \eta X_1$ and $X_i' = X_i/(1-\eta\theta)$. Thus, the Hamiltonian of a free particle can be written as

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} = \frac{1}{2m}\left(P_1' + \eta(1-\eta\theta)X_2'\right)^2 + \frac{1}{2m}\left(P_2' - \eta(1-\eta\theta)X_1'\right)^2 \quad (428)$$

It is worth mentioning that the Hamiltonian (428) corresponds to the Hamiltonian of a particle in the magnetic field $\mathbf{B}(0,0,B)$ ($B = c\eta(1-\eta\theta)/e$, e is the charge of the particle, c is the speed of light) in the noncommutative phase space which is characterized by the relations (425), (426), (427) [117].

4.7. Soccer-ball problem and equivalence principle in six-dimensional noncommutative phase space

Let us generalize the conclusions presented in the previous sections to the case of a six-dimensional (3D configurational and 3D momentum space) noncommutative phase space of a canonical type (49)–(51). The relations (49)–(51) can be written for the coordinates and momenta corresponding to different particles as

$$[X_i^{(a)}, X_j^{(b)}] = i\hbar\delta_{ab}\theta_{ij}^{(a)} \quad (429)$$

$$[X_i^{(a)}, P_j^{(b)}] = i\hbar(\delta_{ab}\delta_{ij} + \delta_{ab}\sigma_{ij}^{(a)}) \quad (430)$$

$$[P_i^{(a)}, P_j^{(b)}] = i\hbar\delta_{ab}\eta_{ij}^{(a)} \quad (431)$$

In the classical limit from (429)–(431) we obtain the following Poisson brackets

$$\{X_i^{(a)}, X_j^{(b)}\} = \delta_{ab}\theta_{ij}^{(a)} \quad (432)$$

$$\{X_i^{(a)}, P_j^{(b)}\} = \delta_{ab}\delta_{ij} + \delta_{ab}\sigma_{ij}^{(a)} \quad (433)$$

$$\{P_i^{(a)}, P_j^{(b)}\} = \delta_{ab}\eta_{ij}^{(a)} \quad (434)$$

where $\theta_{ij}^{(a)}$, $\eta_{ij}^{(a)}$, $\sigma_{ij}^{(a)}$ correspond to the particle labeled by the index a.

For the coordinates and momenta satisfying the relations (432), (434), the symmetrical representation

$$X_i^{(a)} = x_i^{(a)} - \frac{1}{2}\sum_j \theta_{ij}^{(a)} p_j^{(a)} \quad (435)$$

$$P_i^{(a)} = p_i^{(a)} + \frac{1}{2}\sum_j \eta_{ij}^{(a)} x_j^{(a)} \quad (436)$$

is well known. The coordinates and momenta $x_i^{(a)}$, $p_i^{(a)}$ in (435), (436) satisfy the ordinary commutation relations (15), (16). It follows from (435), (436) that

$$\sigma_{ij}^{(a)} = \sum_k \frac{\theta_{ik}^{(a)}\eta_{jk}^{(a)}}{4} \quad (437)$$

(see, for instance, [66, 75]).

The Poisson brackets for the coordinates and momenta of the center-of-mass defined in the traditional way (259), (260)

$$\{X_i^c, X_j^c\} = \theta_{ij}^c \tag{438}$$

$$\{X_i^c, P_j^c\} = \delta_{ij} + \sum_a \mu_a \sigma_{ij}^{(a)} \tag{439}$$

$$\{P_i^c, P_j^c\} = \eta_{ij}^c \tag{440}$$

here

$$\theta_{ij}^c = \sum_a \mu_a^2 \theta_{ij}^{(a)} \tag{441}$$

$$\eta_{ij}^c = \sum_a \eta_{ij}^{(a)} \tag{442}$$

do not reproduce the relations of the noncommutative algebra (49)–(51). We have

$$\sum_a \mu_a \sigma_{ij}^{(a)} = \sum_a \mu_a \sum_k \frac{\theta_{ik}^{(a)} \eta_{ik}^{(a)}}{4} \neq \sum_k \frac{\theta_{ik}^c \eta_{jk}^c}{4} \tag{443}$$

Note that for the coordinates and momenta of the center-of-mass and the relative motion (259), (260), (348), (349) the following relations are satisfied

$$\{X_i^c, \Delta X_j^{(a)}\} = \mu_a \theta_{ij}^{(a)} - \sum_b \mu_b^2 \theta_{ij}^{(b)} \tag{444}$$

$$\{P_i^c, \Delta P_j^{(a)}\} = \eta_{ij}^{(a)} - \mu_a \sum_b \eta_{ij}^{(b)} \tag{445}$$

$$\{\Delta X_i^{(a)}, P_j^{(c)}\} = \sigma_{ij}^{(a)} - \sum_b \mu_b \sigma_{ij}^{(b)} \tag{446}$$

$$\{X_i^c, \Delta P_j^{(a)}\} = \mu_a (\sigma_{ij}^{(a)} - \sum_b \mu_b \sigma_{ij}^{(b)}) \tag{447}$$

Similarly as in the four-dimensional case, considering the parameters of noncommutativity $\theta_{ij}^{(a)}$, $\eta_{ij}^{(a)}$ to be dependent on mass as

$$\theta_{ij}^{(a)} m_a = \gamma_{ij} \tag{448}$$

$$\frac{\eta_{ij}^{(a)}}{m_a} = \alpha_{ij} \tag{449}$$

where γ_{ij}, α_{ij} are constants which are the same for different particles we obtain that the relations (444)–(447) vanish

$$\begin{aligned} &\{X_i^c, \Delta X_j^{(a)}\} = \{P_i^c, \Delta P_j^{(a)}\} = 0 \\ &\{\Delta X_i^{(a)}, P_j^{(c)}\} = \{X_i^c, \Delta P_j^{(a)}\} = 0 \end{aligned} \tag{450}$$

and the parameters $\sigma_{ij}^{(n)}$ are the same for different particles

$$\sigma_{ij}^{(a)} = \sum_k \frac{\gamma_{ik}\alpha_{jk}}{4} = \sum_k \frac{\theta_{ik}^c \eta_{jk}^c}{4} = \sum_k \frac{\theta_{ik}^{(a)}\eta_{jk}^{(a)}}{4} = \sigma_{ij} \tag{451}$$

Thus, the coordinates and momenta of the center-of-mass satisfy the relations of the deformed algebra with the effective parameters of noncommutativity [111]

$$\{X_i^c, X_j^c\} = \theta_{ij}^c \tag{452}$$

$$\{X_i^c, P_j^c\} = \delta_{ij} + \sum_k \frac{\theta_{ik}^c \eta_{jk}^c}{4} \tag{453}$$

$$\{P_i^c, P_j^c\} = \eta_{ij}^c \tag{454}$$

$$\theta_{ij}^c = \frac{\gamma_{ij}}{M} \tag{455}$$

$$\eta_{ij}^c = M\alpha_{ij} \tag{456}$$

Also, if the relations (448), (449) are satisfied the motion of a free particle in the noncommutative phase space does not depend on its mass and a system of free particles with the same initial velocities does not fly away [111]. In the space (432)–(434) the equations of motion of a free particle read

$$\dot{X}_i = \sum_j (\delta_{ij} + \sigma_{ij}) \frac{P_j}{m} \tag{457}$$

$$\dot{P}_i = \sum_j \eta_{ij} \frac{P_j}{m} \tag{458}$$

where m is the mass of the particle. From (457), (458) we have

$$\dot{X}_i(t) = A_{i1}\cos\left(\frac{\tilde{\eta}}{m}t\right) + A_{i2}\sin\left(\frac{\tilde{\eta}}{m}t\right) + A_{i3} \tag{459}$$

$$\tilde{\eta} = \sqrt{\eta_{12}^2 + \eta_{23}^2 + \eta_{31}^2} \tag{460}$$

with A_{ij} being elements of the matrix

$$\hat{A} = (1+\hat{\sigma}) \times \begin{pmatrix} \frac{C_2\eta_{31}\tilde{\eta} - C_1\eta_{12}\eta_{23}}{\eta_{23}^2+\eta_{31}^2} & -\frac{C_1\eta_{31}\tilde{\eta} + C_2\eta_{12}\eta_{23}}{\eta_{23}^2+\eta_{31}^2} & \frac{C_3\eta_{23}}{\eta_{12}} \\ -\frac{C_2\eta_{23}\tilde{\eta} + C_1\eta_{12}\eta_{31}}{\eta_{23}^2+\eta_{31}^2} & \frac{C_1\eta_{23}\tilde{\eta} - C_2\eta_{12}\eta_{31}}{\eta_{23}^2+\eta_{31}^2} & \frac{C_3\eta_{23}}{\eta_{12}} \\ C_1 & C_2 & C_3 \end{pmatrix} \tag{461}$$

The constants C_i are determined by the initial velocities υ_{0i}

$$(1+\hat{\sigma})\hat{B}\hat{C} = \hat{\upsilon}_0$$

$$\hat{B} = \begin{pmatrix} \frac{-\eta_{12}\eta_{23}}{\eta_{23}^2+\eta_{31}^2} & \frac{\eta_{31}\tilde{\eta}}{\eta_{23}^2+\eta_{31}^2} & \frac{\eta_{23}}{\eta_{12}} \\ \frac{-\eta_{12}\eta_{31}}{\eta_{23}^2+\eta_{31}^2} & -\frac{\eta_{23}\tilde{\eta}}{\eta_{23}^2+\eta_{31}^2} & \frac{\eta_{31}}{\eta_{12}} \\ 1 & 0 & 1 \end{pmatrix} \tag{462}$$

$$\hat{C} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \qquad \hat{\upsilon}_0 = \begin{pmatrix} \upsilon_{01} \\ \upsilon_{02} \\ \upsilon_{03} \end{pmatrix} \tag{463}$$

Elements of the matrix $\hat{\sigma}$ are given by (437). Note that the motion of a free particle in the noncommutative phase space depends on its mass. The situation is changed if we consider the relations (448), (449). Due to them we can rewrite (457), (458) as

$$\dot{X}_i = \sum_j (\delta_{ij} + \sigma_{ij}) \frac{P_j}{m} \tag{464}$$

$$\frac{\dot{P}_i}{m} = \sum_j \alpha_{ij} \frac{P_j}{m} \tag{465}$$

It follows from (464), (465) that $X_i(t)$ does not depend on mass. For a system of free particles we have that the velocity of its center-of-mass is the same as the velocities of the particles forming it

$$\dot{X}_i^c(t) = \sum_n \mu_n \dot{X}_i^{(n)}(t) = \sum_n \mu_n \left(A_{i1}^{(n)} \cos\left(\frac{\tilde{\eta}^{(n)}}{m_n} t\right) + A_{i2}^{(n)} \sin\left(\frac{\tilde{\eta}^{(n)}}{m_n} t\right) + A_{i3}^{(n)} \right) =$$
$$A_{i1} \cos\left(\sqrt{\alpha_{12}^2 + \alpha_{23}^2 + \alpha_{31}^2}\, t\right) + A_{i2} \sin\left(\sqrt{\alpha_{12}^2 + \alpha_{23}^2 + \alpha_{31}^2}\, t\right) + A_{i3} = \dot{X}_i^{(n)}(t) \tag{466}$$

Here, we take into account the fact that if the condition (449) holds

$$\frac{\tilde{\eta}^{(n)}}{m_n} = \frac{\sqrt{(\eta_{12}^{(n)})^2 + (\eta_{23}^{(n)})^2 + (\eta_{31}^{(n)})^2}}{m_n} = \sqrt{\alpha_{12}^2 + \alpha_{23}^2 + \alpha_{31}^2} \tag{467}$$

and

$$A_{ij}^{(n)} = A_{ij} \tag{468}$$

The relative velocities are equal to zero

$$\Delta \dot{X}_i(t) = \dot{X}_i^{(n)}(t) - \dot{X}_i^c(t) = 0 \tag{469}$$

Hence, as it is in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$), the system of free particles with the same initial velocities does not fly away.

Also, if the conditions (448), (449) are satisfied the weak equivalence principle is preserved in the noncommutative phase space. For a particle (body) of mass m in a gravitational field $V(\mathbf{X})$, due to the relations (448), (449) we can write

$$H=\frac{P^2}{2m}+mV(\mathbf{X}) \tag{470}$$

$$\dot{X}_i=\sum_j(\delta_{ij}+\sigma_{ij})P'_j+\sum_j\gamma_{ij}\frac{\partial V}{\partial X_j} \tag{471}$$

$$\dot{P}'_i=-\sum_j(\delta_{ij}+\sigma_{ij})\frac{\partial V}{\partial X_j}+\sum_j\alpha_{ij}P'_j \tag{472}$$

where P'_i is given by (405). From (471), (472) we have that $X_i(t)$ and $P'_i(t)$ do not depend on mass, therefore, the weak equivalence principle is preserved.

Hence, due to the relations (448), (449) the coordinates and momenta of the center-of-mass satisfy the noncommutative algebra with the effective parameters of noncommutativity; the motion of the center-of-mass is independent of the relative motion; the trajectory of a free particle is independent of its mass; the weak equivalence principle is recovered in the six-dimensional noncommutative phase space (49)–(51) [111].

4.8. Estimation of parameters of noncommutativity based on studies of Mercury's perihelion shift

Let us consider a particle with mass m in the gravitational field $-k/X$, where k is a constant, $X=\sqrt{\sum_i X_i^2}$, in a noncommutative phase space (49)–(51). The perihelion shift of the orbit of the particle up to the first order in the parameters of noncommutativity reads

$$\Delta\phi_{nc}=2\pi\left(\sqrt{\frac{m^2k}{a^3(1-e^2)^3}}\theta+\frac{2}{e^2}\sqrt{\frac{a^3(1-e^2)^3}{m^2k}}\eta\right) \tag{473}$$

where a is the semi-major axis, e is eccentricity, $\theta=\theta_3$, $\eta=\eta_3$ $(\theta_i=\epsilon_{ijk}\theta_{jk}/2$, $\eta_i=\epsilon_{ijk}\eta_{jk}/2)$ [123].

In [123] comparing the perihelion shift caused by noncommutativity (473) with the observed perihelion shift for the Mercury planet and considering the parameters of noncommutativity of the Mercury planet to be the same as the parameters of noncommutativity of a particle, the upper bound for the minimal length close to the Planck length was obtained, namely $\sqrt{\hbar\theta}\leq 6.3\cdot 10^{-33}$m. This result can be reexamined to a more relevant one, if we take into account the fact that the motion of the Mercury planet in the noncommutative phase space is described by the effective parameters of noncommutativity. Therefore, taking into account the fact that $k = GM_S$ (G is the gravitational constant, M_S is the mass of the Sun) the perihelion shift reads

$$\Delta\phi_{nc} = \Delta\phi_\theta + \Delta\phi_\eta \tag{474}$$

$$\Delta\phi_\theta = 2\pi\sqrt{\frac{GM^2M_S}{a^3(1-e^2)^3}}\tilde{\theta} \tag{475}$$

$$\Delta\phi_\eta = \frac{4\pi}{e^2}\sqrt{\frac{a^3(1-e^2)^3}{GM^2M_S}}\tilde{\eta} \tag{476}$$

where M is the mass of Mercury, $\tilde{\theta}$, $\tilde{\eta}$ are given by (344), (345).

Similarly as was done in the Section 3.6, assuming that

$$|\Delta\phi_{nc}| \leq |\Delta\phi_{obs} - \Delta\phi_{GR}| \tag{477}$$

at 3σ we can write

$$|\Delta\phi_{nc}| \leq 2\pi \cdot 10^{-11} \text{ radians/revolution} \tag{478}$$

Since either of the two contributions $\Delta\phi_\theta$, $\Delta\phi_\eta$ to $\Delta\phi_{nc}$ could be equal to zero, we can write

$$\begin{aligned} &|\Delta\phi_\theta| \leq 2\pi \cdot 10^{-11} \text{ radians/revolution} \\ &|\Delta\phi_\eta| \leq 2\pi \cdot 10^{-11} \text{ radians/revolution} \end{aligned} \tag{479}$$

Thus, using (475), (476), we find

$$\hbar|\tilde{\theta}| \leq 3.6 \cdot 10^{-63} \text{ m}^2 \tag{480}$$

$$\hbar|\tilde{\eta}| \leq 6.5 \cdot 10^{-30} \text{ kg}^2 \text{ m}^2/\text{ s}^2 \tag{481}$$

Let us reexamine the obtained result for the parameters of noncommutativity corresponding to electrons and nucleons. On the basis of (362), (363) we have

$$\begin{aligned} \tilde{\theta} &= \frac{\theta_e m_e}{M} = \frac{\theta_{nuc} m_{nuc}}{M} \\ \tilde{\eta} &= \frac{\eta_e M}{m_e} = \frac{\eta_{nuc} M}{m_{nuc}} \end{aligned} \tag{482}$$

Therefore, using (480), (481) for parameters of noncommutativity corresponding to nucleons we obtain

$$\hbar|\theta_{nuc}| \leq 7.2 \cdot 10^{-13} \text{ m}^2 \tag{483}$$

$$\hbar|\eta_{nuc}| \leq 3.3 \cdot 10^{-80} \text{ kg}^2 \text{ m}^2/\text{ s}^2 \tag{484}$$

and for parameters of noncommutativity of electrons we find [111]

$$\hbar|\theta_e| \leq 1.3 \cdot 10^{-9} \text{ m}^2 \tag{485}$$

$$\hbar|\eta_e| \leq 1.8 \cdot 10^{-83} \text{ kg}^2 \text{ m}^2/\text{ s}^2 \tag{486}$$

For the constants γ, α which are presented in the relation (362), (363) we have

$$|\gamma| \leq 1.1 \cdot 10^{-5} \text{ s} = 2.1 \cdot 10^{38} \text{ T}_P \tag{487}$$

$$|\alpha| \leq 1.9 \cdot 10^{-19} \text{ s}^{-1} = 10^{-62} \text{ T}_P^{-1} \tag{488}$$

here T_P is the Planck time [111].

The result (483) is in agreement with the result obtained on the basis of the studies of neutrons in gravitational well [124]. The inequality (485) does not impose any strong restriction on the value of the parameter of the coordinate noncommutativity. This is due to a reduction in the effective parameter of coordinate noncommutativity with respect to the parameters of noncommutativity corresponding to the elementary particles (346).

The results for the parameter of momentum noncommutativity (484), (486) are quite strong. The upper bound (484) is 13 orders less than that obtained examining neutrons in a gravitational quantum well [125]. The result (486) is 17 orders less than that obtained on the basis of the studies of the effect of noncommutativity on the hyperfine structure of the hydrogen atom [126]. On the basis of (486) we obtain the following upper bound on the momentum scale

$$\sqrt{\hbar|\eta_e|} \leq 4.2 \cdot 10^{-42} \text{ kg} \cdot \text{ m}/\text{ s} = 6.5 \cdot 10^{-43} \text{ E}_P/\text{ c} \tag{489}$$

where E_P is the Planck energy.

From the Heisenberg uncertainty relation we can write $\Delta P \geq \hbar/2\Delta X$. For the distance which corresponds to the diameter of the observable universe $8.8 \cdot 10^{26}$m [127] we obtain $\Delta P \geq 6 \cdot 10^{-62}$ kg$\cdot$ m/ s. The result (489) is many orders greater than this value. We have $\sqrt{\hbar|\eta_e|}/\Delta P = 7 \cdot 10^{19}$ [111].

4.9. Effect of noncommutativity on the Sun-Earth-Moon system and the weak equivalence principle

According to the equivalence principle the free fall accelerations of the Earth and the Moon toward the Sun in the case when the bodies are at the same distance to the source of gravity are the same. On the basis of the Lunar laser ranging experiment it has been obtained that the equivalence principle holds with the accuracy

$$\frac{\Delta a}{a} = \frac{2(a_E - a_M)}{a_E + a_M} = (-0.8 \pm 1.3) \cdot 10^{-13} \tag{490}$$

where a_E, a_M are the free fall accelerations of the Earth and the Moon toward the Sun in the case when the Earth and the Moon are at the same distance from the Sun [108]. This result can be used to estimate the precision with which the conditions on the parameters of noncommutativity (360), (361) are satisfied.

Hence, let us study the influence of noncommutativity of coordinates and noncommutativity of momenta on the Sun-Earth-Moon system and find corrections caused by noncommutativity on the Eötvös parameter (490). Assuming that the influence of the relative motion of particles which form the macroscopic bodies on the motion of their center-of-mass is not significant, we consider the following Hamiltonian

$$H=\frac{(\mathbf{P}^E)^2}{2m_E}+\frac{(\mathbf{P}^M)^2}{2m_M}-G\frac{m_E m_S}{R_{ES}}-G\frac{m_M m_S}{R_{MS}}-G\frac{m_M m_E}{R_{EM}} \tag{491}$$

where G is the gravitational constant, m_S, m_E, m_M are masses of the Sun, the Earth and the Moon, R_{ES}, R_{MS}, R_{EM} are the distances between the Earth and the Sun, the Earth and the Moon, the Moon and the Sun. If we chose the coordinate system with the origin at the Sun we have

$$\begin{aligned}
R_{ES}&=\sqrt{(X_1^E)^2+(X_2^E)^2}\\
R_{MS}&=\sqrt{(X_1^M)^2+(X_2^M)^2}\\
R_{EM}&=\sqrt{(X_1^E-X_1^M)^2+(X_2^E-X_2^M)^2}
\end{aligned} \tag{492}$$

where X_i^E, X_i^M are coordinates of the center-of-mass of the Earth and the Moon which satisfy the relations of the noncommutative algebra (341)–(343) with the parameters of noncommutativity θ_E, η_E and θ_M, η_M, respectively. Note that in (491) we consider the inertial masses of the bodies (the masses in the first two terms) to be equal to the gravitational masses (masses in the last three terms). The equations of motion are the following

$$\begin{aligned}
\dot{X}_1^E&=\frac{P_1^E}{m_E}+\theta_E\frac{Gm_E m_S X_2^E}{R_{ES}^3}+\theta_E\frac{Gm_E m_M(X_2^E-X_2^M)}{R_{EM}^3}\\
\dot{X}_2^E&=\frac{P_2^E}{m_E}-\theta_E\frac{Gm_E m_S X_1^E}{R_{ES}^3}-\theta_E\frac{Gm_E m_M(X_1^E-X_1^M)}{R_{EM}^3}\\
\dot{P}_1^E&=\eta_E\frac{P_2^E}{m_E}-\frac{Gm_E m_S X_1^E}{R_{ES}^3}-\frac{Gm_E m_M(X_1^E-X_1^M)}{R_{EM}^3}\\
\dot{P}_2^E&=-\eta_E\frac{P_1^E}{m_E}-\frac{Gm_E m_S X_2^E}{R_{ES}^3}-\frac{Gm_E m_M(X_2^E-X_2^M)}{R_{EM}^3}\\
\dot{X}_1^M&=\frac{P_1^M}{m_M}+\theta_M\frac{Gm_M m_S X_2^M}{R_{MS}^3}-\theta_M\frac{Gm_E m_M(X_2^E-X_2^M)}{R_{EM}^3}\\
\dot{X}_2^M&=\frac{P_2^M}{m_M}-\theta_M\frac{Gm_M m_S X_1^E}{R_{MS}^3}+\theta_M\frac{Gm_E m_M(X_1^E-X_1^M)}{R_{EM}^3}\\
\dot{P}_1^M&=\eta_M\frac{P_2^M}{m_M}-\frac{Gm_M m_S X_1^M}{R_{MS}^3}+\frac{Gm_E m_M(X_1^E-X_1^M)}{R_{EM}^3}\\
\dot{P}_2^M&=-\eta_M\frac{P_1^M}{m_M}-\frac{Gm_M m_S X_2^M}{R_{MS}^3}+\frac{Gm_E m_M(X_2^E-X_2^M)}{R_{EM}^3}
\end{aligned} \tag{493}$$

Let us choose the X_1 axis to be perpendicular to $\mathbf{R}_{EM}(X_1^E - X_1^M, X_2^E - X_2^M)$ and to pass through the middle of the vector $\mathbf{R}_{EM}$, the X_2 axis to be parallel to $\mathbf{R}_{EM}$ (the origin of the frame of references is chosen to be at the Sun). Hence, if the Moon and the Earth are at the same distance to the source of gravity $R_{MS} = R_{ES} = R$ we have

$$\begin{aligned} X_1^E &= X_1^M = R\sqrt{1 - \frac{R_{EM}^2}{4R^2}} \\ X_2^E &= -X_2^M = \frac{R_{EM}}{2} \end{aligned} \tag{494}$$

and, taking into account that $R_{EM}/R \sim 10^{-3}$, we can write $X_1^E \simeq R$. Therefore, from (493) we obtain the following expressions for free fall accelerations of the Moon and the Earth toward the Sun

$$a_E = \ddot{X}_1^E = -\frac{Gm_S}{R^2} + \eta_E \frac{\upsilon_E}{m_E} + \theta_E \frac{Gm_S m_E \upsilon_E}{R^3}\left(1 - \frac{3R_{EM}}{2\upsilon_E R^2}(\mathbf{R}_{ES} \cdot \dot{\mathbf{R}}_{ES})\right) \tag{495}$$

$$a_M = \ddot{X}_1^M = -\frac{Gm_S}{R^2} + \eta_M \frac{\upsilon_E}{m_M} + \theta_M \frac{Gm_S m_M \upsilon_E}{R^3}\left(1 + \frac{3R_{EM}}{2\upsilon_E R^2}(\mathbf{R}_{MS} \cdot \dot{\mathbf{R}}_{MS})\right) \tag{496}$$

Writing (495), (496) we take into account that $\dot{X}_1^E = 0$, $\dot{X}_2^E = \dot{X}_2^M = \upsilon_E$, $\dot{X}_1^M = \upsilon_M$, υ_E, υ_M are the orbital velocities of the Earth and the Moon. Due to the relations $R_{EM}/R \sim 10^{-3}$, $\upsilon_M/\upsilon_E \sim 10^{-2}$ the last terms in (495), (496) can be neglected, and the Eötvös parameter is the following [128]

$$\frac{\Delta a}{a} = \frac{2(a_E - a_M)}{a_E + a_M} = \frac{\Delta a^\eta}{a} + \frac{\Delta a^\theta}{a} \tag{497}$$

where $\Delta a^\eta / a$ denotes correction to the Eötvös parameter caused by the noncommutativity of the momenta

$$\frac{\Delta a^\eta}{a} = \frac{\upsilon_E R^2}{G m_S}\left(\frac{\eta_E}{m_E} - \frac{\eta_M}{m_M}\right) \tag{498}$$

and $\Delta a^\theta / a$ is correction caused by the noncommutativity of coordinates

$$\frac{\Delta a^\theta}{a} = \frac{\upsilon_E}{R}(\theta_E m_E - \theta_M m_M) \tag{499}$$

It is important to mention that due to the noncommutativity the Eötvös parameter (497) is not equal to zero, even if the inertial masses of the bodies are equal to the gravitational masses. The correction to the Eötvös-parameter caused by the noncommutativity of coordinates is proportional to $\gamma_E - \gamma_M$ and the correction caused by the noncommutativity of momenta is proportional to $\alpha_E - \alpha_M$, where

$$\gamma_E = \theta_E m_E, \quad \gamma_M = \theta_M m_M$$
$$\alpha_E = \frac{\eta_E}{m_E}, \quad \alpha_M = \frac{\eta_M}{m_M} \tag{500}$$

Note that the expression (497) is equal to zero and the weak equivalence principle is recovered, if the conditions (360), (361) are satisfied, namely if

$$\gamma_E = \gamma_M = \gamma$$
$$\alpha_E = \alpha_M = \alpha \tag{501}$$

On the basis of the results (490), (497) we can find the upper bound for

$$\Delta\alpha = \alpha_E - \alpha_M$$
$$\Delta\gamma = \gamma_E - \gamma_M \tag{502}$$

and therefore, estimate the precision with which the proposed conditions (360), (361) hold.

Assuming that corrections to the Eötvös-parameter caused by the noncommutativity of coordinates and the noncommutativity of momenta (497) are less than the experimental results for the limits on violation of the weak equivalence principle (490) we can write the following inequality

$$\left|\frac{\Delta a^\theta + \Delta a^\eta}{a}\right| \leq 2.1 \cdot 10^{-13} \tag{503}$$

where $2.1 \cdot 10^{-13}$ is the largest value of $|\Delta a|/|a|$ given by (490) [108]. To estimate the orders of $\Delta\alpha$, $\Delta\gamma$ it is sufficient to consider the inequalities

$$\left|\frac{\Delta a^\theta}{a}\right| \leq 2.1 \cdot 10^{-13} \tag{504}$$

$$\left|\frac{\Delta a^\eta}{a}\right| \leq 2.1 \cdot 10^{-13} \tag{505}$$

from which we find [128]

$$\Delta\alpha \leq 10^{-20} \text{ s}^{-1} \tag{506}$$

$$\Delta\gamma \leq 10^{-6} \text{ s} \tag{507}$$

Let us analyze the obtained results. Using (488) we have $\Delta\alpha/\alpha \leq 5$. Assuming that the minimal length corresponding to the electron corresponds to the Planck length $\sqrt{\hbar\,|\,\theta_e\,|} = l_P$, we obtain

$$\gamma = \frac{m_e L_p^2}{\hbar} = 4.2 \cdot 10^{-23} T_p = 2.3 \cdot 10^{-66} \text{ s} \tag{508}$$

where T_P is the Planck time. Hence, the results for $\Delta\alpha$, $\Delta\gamma$ (506), (507) are not strong. Results with higher accuracy are needed to find stronger restrictions on these values.

5. Composite system in noncommutative phase space of canonical type with rotational and time-reversal symmetries

In this chapter problems of many particles are considered in the frame of a rotationally and time reversal invariant noncommutative algebra of a canonical type constructed on the basis of the idea to generalize the parameters of noncommutativity to tensors. The tensors are considered to be dependent on additional momenta governed by harmonic oscillators (see Section 2.3). We show that the relation of tensors of noncommutativity with mass opens the possibility to solve the problem of a macroscopic body and the problem of violation of the weak equivalence principle in the rotationally and time reversal invariant noncommutative phase space of a canonical type.

The chapter is organized as follows. In the Section 5.1 the Hamiltonian of a system in a noncommutative phase space with preserved rotational and time-reversal symmetries is considered. Section 5.2 is devoted to studies of commutation relations for coordinates and momenta of the center-of-mass of a composite system in a noncommutative phase space. In Section 5.3 the motion of a particle (body) in a uniform gravitational field is considered and the weak equivalence principle is examined. In Section 5.4 quantum and classical equations of motion of a particle in a non-uniform gravitational field are presented and the implementation of the weak equivalence principle is studied.

5.1. Hamiltonian in noncommutative phase space with rotational and time reversal symmetries

Examining a system in a noncommutative phase space of a canonical type with preserved rotational and time reversal symmetries (118)–(120), the total Hamiltonian has to be considered

$$H = H_s + H^a_{osc} + H^b_{osc} \tag{509}$$

which is the sum of the Hamiltonian of the system H_s and the Hamiltonians of harmonic oscillators H^a_{osc}, H^b_{osc} (108), (109). This is by because of involving additional coordinates and additional momenta for the construction of tensors of noncommutativity (579), (112).

It is convenient to introduce

$$H_0=\langle H_s\rangle_{ab}+H^a_{osc}+H^b_{osc} \tag{510}$$

$$\Delta H=H-H_0=H_s-\langle H_s\rangle_{ab} \tag{511}$$

and rewrite the Hamiltonian (509) as

$$H=H_0+\Delta H \tag{512}$$

In (510), (511) $\langle ...\rangle_{ab}$ denotes averaging over the degrees of freedom of harmonic oscillators in the ground states[4]

$$\langle ...\rangle_{ab}=\langle\psi^a_{0,0,0}\psi^b_{0,0,0}|...|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle \tag{513}$$

The functions $\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$ are well known and correspond to the ground states of three-dimensional harmonic oscillators in the ordinary space ($\theta_{ij}=\eta_{ij}=0$).

Up to the second order in ΔH the corrections to the spectrum of the total Hamiltonian (512) caused by the term ΔH vanish [129]. In the first order of the perturbation theory in ΔH we have

$$\Delta E^{(1)}=\langle\psi^{(0)}|\Delta H|\psi^{(0)}\rangle \tag{514}$$

where $\psi^{(0)}$ are the eigenstates of H_0. Note that

$$[\langle H_s\rangle_{ab},H^a_{osc}+H^b_{osc}]=0 \tag{515}$$

Hence, the eigenstates of H_0 can be written in the following form

$$\psi^{(0)}_{\{n_s\},\{0\},\{0\}}=\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0} \tag{516}$$

where $\{n_s\}$ are quantum numbers, $\psi^s_{\{n_s\}}$ are the eigenstates of $\langle H_s\rangle_{ab}$. The eigenvalues of H_0 read

$$E^{(0)}_{\{n_s\}}=E^s_{\{n_s\}}+3\hbar\omega_{osc} \tag{517}$$

Here, we take into account the fact that the oscillators H^a_{osc}, H^b_{osc} are in the ground states. Hence, in the first order of the perturbation theory we have

$$\Delta E^{(1)}=\langle\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}|\Delta H|\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle=$$
$$\langle\psi^s_{\{n_s\}}|\langle H_s\rangle_{ab}-\langle H_s\rangle_{ab}|\psi^s_{\{n_s\}}\rangle=0 \tag{518}$$

In the second order of the perturbation theory in ΔH the corrections read

$$\Delta E^{(2)}=\sum_{\{n'_s\},\{n^a\},\{n^b\}}\frac{\left|\left\langle\psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}}|\Delta H|\psi^{(0)}_{\{n_s\},\{0\},\{0\}}\right\rangle\right|^2}{E^s_{\{n'_s\}}-E^s_{\{n_s\}}-\hbar\omega_{osc}(n^a_1+n^a_2+n^a_3+n^b_1+n^b_2+n^b_3)} \tag{519}$$

4. The frequency ω_{osc} of the harmonic oscillators H^a_{osc}, H^b_{osc} is large, therefore, oscillators being in the ground states remain in them [80]

In (519) the sets of numbers $\{n'_s\}$, $\{n^a\}$, $\{n^b\}$ and $\{n_s\}$,$\{0\}$, $\{0\}$ do not coincide. Therefore, for all terms in (519) in the denominator there is a term proportional to the frequency ω_{osc}. The average values

$$\left\langle \psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}} |\Delta H| \psi^{(0)}_{\{n_s\},\{0\},\{0\}} \right\rangle \tag{520}$$

do not depend on ω_{osc} because of the relation (110). It is worth remembering that the frequency ω_{osc} is large. Therefore, for $\omega_{osc} \to \infty$ we find

$$\lim_{\omega_{osc}\to\infty} \Delta E^{(2)} = 0 \tag{521}$$

Hence, up to the second order in ΔH, the corrections to the spectrum of the total Hamiltonian (512) vanish. Therefore, up to the second order in ΔH we can consider the Hamiltonian given by (510). This conclusion will be used in the next sections for studies of motion of a macroscopic body in the rotationally and time reversal invariant noncommutative phase space. At the end of this section we would like to note that on the basis of this conclusion the spectrum of free particle, the spectrum of the harmonic oscillator, the eigenvalues of the operator of squared length can be easily found and the expressions for the minimal length and the minimum momentum can be obtained up to the second order in the parameters of noncommutativity.

For a free particle with mass m we have

$$H_s = \sum_i \frac{P_i^2}{2m} = \frac{p^2}{2m} - \frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} \tag{522}$$

$$\langle H_s\rangle_{ab} = \frac{p^2}{2m} + \frac{\langle\eta^2\rangle x^2}{12m} \tag{523}$$

$$\Delta H = -\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{\langle\eta^2\rangle x^2}{12m} \tag{524}$$

where we use the representation

$$X_i = x_i - \frac{1}{2}\theta_{ij}p_j \tag{525}$$

$$P_i = p_i + \frac{1}{2}\eta_{ij}x_j \tag{526}$$

(coordinates and momenta x_i, p_i satisfy the ordinary commutation relations) and take into account the following relations

$$\langle\psi^a_{0,0,0}|\eta_i|\psi^a_{0,0,0}\rangle = 0 \tag{527}$$

$$\langle\eta^2\rangle = \sum_i \langle\eta_i^2\rangle = \sum_i \frac{c_\eta^2}{\hbar^2}\langle\psi^b_{0,0,0}|(p_i^b)^2|\psi^b_{0,0,0}\rangle = \frac{3c_\eta^2}{2l_P^2} \tag{528}$$

The components of the vector $\boldsymbol{\eta}=(\eta_1,\eta_2,\eta_3)$ read

$$\eta_i=\frac{1}{2}\sum_{jk}\varepsilon_{ijk}\eta_{jk} \tag{529}$$

Taking into account the expression for ΔH (524) we have that up to the second order in ΔH or up to the second order in the parameter of momentum noncommutativity, the free particle is described by the Hamiltonian (523) and its spectrum reads

$$E_{n_1,n_2,n_3}=\sqrt{\frac{\hbar^2\langle\eta^2\rangle}{6m^2}}\left(n_1+n_2+n_3+\frac{3}{2}\right) \tag{530}$$

where n_i $(i=(1,2,3))$ are quantum numbers $n_i=0,1,2...$. Hence, the noncommutativity of momenta causes quantization of the energy of the free particle. The energy levels of the free particle correspond to the energy levels of the tree-dimensional harmonic oscillator with the frequency determined by the parameter of momentum noncommutativity [130].

For a three-dimensional harmonic oscillator with mass m and frequency ω we have

$$H_s=\sum_i\frac{P_i^2}{2m}+\sum_i\frac{m\omega^2X_i^2}{2}=\frac{p^2}{2m}+\frac{m\omega^2x^2}{2}-\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m}-\frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}\times\mathbf{p}])}{2}+\frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}+\frac{m\omega^2[\boldsymbol{\theta}\times\mathbf{p}]^2}{8} \tag{531}$$

$$\langle H_s\rangle_{ab}=\left(\frac{1}{2m}+\frac{m\omega^2\langle\theta^2\rangle}{12}\right)p^2+\left(\frac{m\omega^2}{2}+\frac{\langle\eta^2\rangle}{12m}\right)x^2 \tag{532}$$

$$\Delta H=-\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}\times\mathbf{p}])}{2m}-\frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}\times\mathbf{p}])}{2}+\frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}+\frac{m\omega^2[\boldsymbol{\theta}\times\mathbf{x}]^2}{8}-\frac{m\omega^2\langle\theta^2\rangle}{12}p^2-\frac{\langle\eta^2\rangle}{12m}x^2 \tag{533}$$

where we use (527), (528) and

$$\langle\psi^a_{0,0,0}|\theta_i|\psi^a_{0,0,0}\rangle=0 \tag{534}$$

$$\langle\theta^2\rangle=\sum_i\langle\theta_i^2\rangle=\sum_i\frac{c_\theta^2}{\hbar^2}\langle\psi^a_{0,0,0}|(p_i^a)^2|\psi^a_{0,0,0}\rangle=\frac{3c_\theta^2}{2l_P^2} \tag{535}$$

$$\boldsymbol{\theta}=(\theta_1,\theta_2,\theta_3),\quad \theta_i=\frac{1}{2}\sum_{jk}\varepsilon_{ijk}\theta_{jk} \tag{536}$$

Thus, up to the second order in the parameters of noncommutativity we have the following energy levels

$$E_{n_1,n_2,n_3}=\hbar\sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)}\left(n_1+n_2+n_3+\frac{3}{2}\right) \quad (537)$$

where n_i $(i=(1,2,3))$ are quantum numbers, $n_i=0,1,2...$, [130, 131]. Note that the expression (537) corresponds to the spectrum of the harmonic oscillator in the ordinary space with an effective mass and an effective frequency

$$m_{eff}=\frac{6m}{6+m^2\omega^2\langle\theta^2\rangle} \quad (538)$$

$$\omega_{eff}=\sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)} \quad (539)$$

In the limit $\langle\theta^2\rangle\to 0$, $\langle\eta^2\rangle\to 0$ from (538), (539) we obtain $m_{eff}=m$, $\omega_{eff}=\omega$. Hence, the expression (537) reduces to the spectrum of the harmonic oscillator in the ordinary space. It is worth mentioning that the problem of harmonic oscillator is well studied in the frame of different noncommutative algebras of a canonical type [132–136, 58, 137–139, 64, 65, 140–142, 48, 62, 143–147].

On the basis of these results we can also write eigenvalues of the squared length operator

$$\mathbf{Q}^2=c_1^2\sum_i P_i^2+c_2^2\sum_i X_i^2 \quad (540)$$

(where c_1 and c_2 are constants). Up to the second order in the parameters of noncommutativity the eigenvalues of the operator $\mathbf{Q}^2$ are

$$q^2_{n_1,n_2,n_3}=\hbar\sqrt{\left(2\beta^2+\frac{\alpha^2\langle\eta^2\rangle}{3}\right)\left(2\alpha^2+\frac{\beta^2\langle\theta^2\rangle}{3}\right)}\left(n_1+n_2+n_3+\frac{3}{2}\right) \quad (541)$$

$n_i=0,1,2...$. For $c_1=0$, $c_2=1$ the eigenvalues of the operator $\mathbf{Q}^2=\sum_{i=1}^3 X_i^2=\mathbf{R}^2$ read

$$r^2_{n_1,n_2,n_3}=\sqrt{\frac{2\hbar^2\langle\theta^2\rangle}{3}}\left(n_1+n_2+n_3+\frac{3}{2}\right) \quad (542)$$

where $n_i=0,1,2....$ Thus, the minimal length is defined as

$$r_{min}=\sqrt{r^2_{0,0,0}}=\sqrt{\frac{3\hbar^2\langle\theta^2\rangle}{2}} \quad (543)$$

Similarly the eigenvalues of the operator $\mathbf{Q}^2$ in the case when $c_1=1$, $c_2=0$ $\mathbf{Q}^2=\sum_{i=1}^3 P_i^2=\mathbf{P}^2$ are

$$p^2_{n_1,n_2,n_3}=\sqrt{\frac{2\hbar^2\langle\eta^2\rangle}{3}}\left(n_1+n_2+n_3+\frac{3}{2}\right) \quad (544)$$

$n_i=0,1,2...$ and the minimum momentum is defined as

$$p_{min}=\sqrt{p^2_{0,0,0}}=\sqrt[4]{\frac{3\hbar^2\langle\eta^2\rangle}{2}} \tag{545}$$

5.2. Composite system in the frame of rotationally and time reversal invariant noncommutative algebra of canonical type

Considering a system of N particles in a noncommutative phase space with rotational and time reversal symmetries one has to generalize the relations of the noncommutative algebra (118)–(120) to the case of coordinates and momenta of different particles. In a general case when the coordinates and momenta of different particles satisfy the relations of the noncommutative algebra with different tensors of noncommutativity. Assuming that the coordinates and momenta corresponding to different particles commute, we can write the following relations

$$[X_i^{(n)},X_j^{(m)}]=i\hbar\delta_{mn}\theta_{ij}^{(n)} \tag{546}$$

$$[X_i^{(n)},P_j^{(m)}]=i\hbar\delta_{mn}\left(\delta_{ij}+\sum_k\frac{\theta_{ik}^{(n)}\eta_{jk}^{(m)}}{4}\right) \tag{547}$$

$$[P_i^{(n)},P_j^{(m)}]=i\hbar\delta_{mn}\eta_{ij}^{(n)} \tag{548}$$

where m,n label the particles, and $\theta_{ij}^{(n)}$, $\eta_{ij}^{(n)}$ are the tensors of noncommutativity, corresponding to the particle labeled by index n.

Let us consider the tensors of noncommutativity to be dependent on mass. Let us generalize (579), (112) as

$$\theta_{ij}^{(n)}=\frac{c_\theta^{(n)}}{\hbar}\sum_k\varepsilon_{ijk}p_k^a \tag{549}$$

$$\eta_{ij}^{(n)}=\frac{c_\eta^{(n)}}{\hbar}\sum_k\varepsilon_{ijk}p_k^b \tag{550}$$

Similarly as in the case of the noncommutative algebra of a canonical type we consider the tensor of coordinate noncommutativity to be inversely proportional to mass and the tensor of momentum noncommutativity to be proportional to mass. Namely, we assume that

$$c_\theta^{(n)}=\frac{\tilde\gamma}{m_n} \tag{551}$$

$$c_\eta^{(n)}=\tilde\alpha m_n \tag{552}$$

where $\tilde\gamma$, $\tilde\alpha$ are constants which do not depend on mass [129]. Additional momenta p_i^a, p_i^b are introduced to construct the tensors of noncommutativity. These momenta are responsible for the phase space noncommutativity. Different particles correspond to the same noncommutative phase space. Therefore, in (549), (550) we consider additional momenta p_i^a, p_i^b to be the same for different particles. At the same time particles with different masses feel different effects of noncommutativity due to the relations (551), (552).

If the tensors of noncommutativity (549), (550) depend on mass as (551), (552), the coordinates and momenta of the center-of-mass (259), (260) satisfy the noncommutative algebra (546)–(548)

$$[X_i^c, X_j^c] = i\hbar\theta_{ij}^c \tag{553}$$

$$[P_i^c, P_j^c] = i\hbar\eta_{ij}^c \tag{554}$$

$$[X_i^c, P_j^c] = i\hbar(\delta_{ij} + \sum_k \frac{\theta_{ik}^c \eta_{jk}^c}{4}) \tag{555}$$

with the effective tensors of noncommutativity [129]

$$\theta_{ij}^c = \frac{\tilde{\gamma}}{\hbar M} \sum_k \varepsilon_{ijk} p_k^a \tag{556}$$

$$\eta_{ij}^c = \frac{\tilde{\alpha}\hbar M}{l_P^2} \sum_k \varepsilon_{ijk} p_k^b \tag{557}$$

and can be represented as

$$X_i^c = \sum_n \mu_n (x_i^{(n)} - \frac{1}{2}\theta_{ij}^{(n)} p_j^{(n)}) = x_i^c - \frac{1}{2}\theta_{ij}^c p_j^c \tag{558}$$

$$P_i^c = \sum_n (p_i^{(n)} + \frac{1}{2}\eta_{ij}^{(n)} x_j^{(n)}) = p_i^c + \frac{1}{2}\eta_{ij}^c x_j^c \tag{559}$$

where

$$x_i^c = \sum_n \mu_n x_i^{(n)} \tag{560}$$

$$p_i^c = \sum_n p_i^{(n)} \tag{561}$$

Note that the following relations hold $[x_i^c, x_j^c] = [p_i^c, p_j^c] = 0$, $[x_i^c, p_j^c] = i\hbar\delta_{ij}$. Also, if the conditions (551), (552) are satisfied, using the definition of the relative coordinates and relative momenta $\Delta\mathbf{X}^{(n)}$, $\Delta\mathbf{P}^{(n)}$ (348), (349) and taking into account (525), (526), we can write the representation for coordinates and momenta of the relative motion

$$\Delta X_i^{(n)} = \Delta x_i^{(n)} - \frac{1}{2}\theta_{ij}^{(n)} \Delta p_j^{(n)} \tag{562}$$

$$\Delta P_i^{(n)} = \Delta p_i^{(n)} + \frac{1}{2}\eta_{ij}^{(n)} \Delta x_j^{(n)} \tag{563}$$

with

$$\Delta x_i^{(n)} = x_i^{(n)} - x_i^c \tag{564}$$

$$\Delta p_i^{(n)} = p_i^{(n)} - \mu_n p_i^c \tag{565}$$

In the next section these results will be used in studies of motion of a body in a gravitational field in the noncommutative phase space with time reversal and rotational symmetries.

5.3. Motion in uniform gravitational field and the weak equivalence principle

Let us consider a particle of mass m in a uniform gravitational field in the noncommutative phase space with preserved rotational and time reversal symmetries (118)–(120). Choosing for convenience the X_1 axis to be directed along the direction of the field, we can write the following Hamiltonian

$$H_p = \frac{\mathbf{P}^2}{2m} + mgX_1 \tag{566}$$

where X_i, P_i satisfy the relations (118)–(120). The total Hamiltonian reads $H = H_p + H^a_{osc} + H^b_{osc}$, and using the representation (525), (526) it can be rewritten as follows

$$H = \frac{\mathbf{p}^2}{2m} + mgx_1 - \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1 + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} + H^a_{osc} + H^b_{osc} \tag{567}$$

where $\mathbf{L} = [\mathbf{x}\times\mathbf{p}]$.

Taking into account (510), (511), (527), (528), (534)–(536), for a particle in a uniform gravitational filed we have [122]

$$H_0 = \frac{\mathbf{p}^2}{2m} + mgx_1 + \frac{\langle\eta^2\rangle\mathbf{x}^2}{12m} + H^a_{osc} + H^b_{osc} \tag{568}$$

$$\Delta H = -\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1 + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{\langle\eta^2\rangle\mathbf{x}^2}{12m} \tag{569}$$

As has been shown in Section 5.1 up to the second order in ΔH we can study the Hamiltonian H_0. Analyzing the expression (569) we have that up to the second order in the parameters of noncommutativity a particle in a uniform gravitational field is described by the Hamiltonian (568). The coordinates and momenta x_i, p_i satisfy the ordinary commutation relations, therefore, we obtain the following equations of motion

$$\begin{aligned} \dot{x}_i &= \frac{p_i}{m} \\ \dot{p}_i &= -mg\delta_{i,1} - \frac{\langle\eta^2\rangle x_i}{6m} \end{aligned} \tag{570}$$

The solutions of these equations with the initial conditions $x_i(0)=x_{0i}$, $\dot{x}_i(0)=\upsilon_{0i}$ read

$$x_i(t)=\left(x_{0i}+6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right)+$$
$$\upsilon_{0i}\sqrt{\frac{6m^2}{\langle\eta^2\rangle}}\sin\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right)-6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i} \tag{571}$$

For $\langle\eta^2\rangle\to 0$ from (571) we find the well known result $x_i(t)=\delta_{1,i}gt^2/2+x_{0i}$.

Note that up to the second order in the parameters of noncommutativity the motion of a particle in a uniform field is affected only by the momentum noncommutativity. The trajectory of the particle (571) depends on its mass, therefore, the noncommutativity of momenta causes violation of the weak equivalence principle.

It is important to note that if the condition (552) is satisfied, using (528), we can write

$$\frac{\langle\eta^2\rangle}{m^2}=\frac{3\tilde{\alpha}^2}{2l_P^2}=B \tag{572}$$

where the constant B does not depend on mass. Therefore, the trajectory of the particle reads

$$x_i(t)=\left(x_{0i}+\frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right)+\upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right)-\frac{6g}{B}\delta_{1,i} \tag{573}$$

Hence, due to the condition (552) the motion of a particle in a uniform gravitational field does not depend on its mass and the weak equivalence principle is preserved [122].

In a more general case of motion of a composite system (macroscopic body) of mass M in a uniform field we have the following Hamiltonian

$$H_s=H_{cm}+H_{rel},\quad H_{cm}=\frac{(\mathbf{P}^c)^2}{2M}+MgX_1^{(c)} \tag{574}$$

where $\mathbf{X}^{(\mathbf{c})}$, $\mathbf{P}^c$ are the coordinates and momenta of the center-of-mass of the system, the term H_{rel} describes the relative motion.

If the relations (551), (552) are satisfied the coordinates and the momenta of the center-of-mass and the coordinates and the momenta of the relative motion can be represented as (558), (559), (562), (563). Therefore, the Hamiltonian H_0 can be written as

$$\begin{gathered}H_0=\langle H_{cm}\rangle_{ab}+\langle H_{rel}\rangle_{ab}+H_{osc}^{(a)}+H_{osc}^{(b)}\\ \langle H_{cm}\rangle_{ab}=\frac{(\mathbf{p}^c)^2}{2M}+Mgx_1^c+\frac{\langle(\eta^c)^2\rangle(\mathbf{x}^c)^2}{12M}\end{gathered} \tag{575}$$

The term $\langle H_{rel}\rangle_{ab}$ depends on $\Delta x_i^{(n)}$, $\Delta p_i^{(n)}$. The coordinates and momenta $\Delta x_i^{(n)}$, $\Delta p_i^{(n)}$ are given by (564), (565) and commute with x_c^i and p_i^c (560), (561). The operators x_c^i, p_i^c, $\Delta x_i^{(n)}$, $\Delta p_i^{(n)}$ also commute with $\tilde{a}_i$, $\tilde{b}_i$,$\tilde{p}_i^a$, $\tilde{p}_i^b$. Therefore, the trajectory of the center-of-mass of a composite system in a uniform gravitation field reads

$$x_i^c(t) = \left(x_{0i}^c + 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) + \upsilon_{0i}^c\sqrt{\frac{6M^2}{\langle(\eta^c)^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) - 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i} \tag{576}$$

Due to the relation (552) we have

$$\langle(\eta^c)^2\rangle = \frac{3\tilde{\alpha}^2 M^2}{2l_P^2} = BM' \tag{577}$$

and the trajectory of the center-of-mass can be rewritten as [122]

$$x_i^c(t) = \left(x_{0i}^c + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i} \tag{578}$$

Comparing (578) with (573) we can see that the motion of a macroscopic body in a gravitational field is the same as the motion of a particle. The expression (578) does not depend on the mass of the body and its composition. Hence, the weak equivalence principle is recovered in the noncommutative phase space with the preserved rotational and time reversal symmetries. This conclusion can be generalized for the case of a non-uniform gravitational field and it will be done in the next section.

5.4. Motion in non-uniform gravitational field and the weak equivalence principle

For a particle of mass m in a nonuniform gravitational field we have the following Hamiltonian

$$H_p = \frac{P^2}{2m} - \frac{mk}{X} \tag{579}$$

where $X = |\mathbf{X}| = \sqrt{\sum_i X_i^2}$, k is a constant (in a gravitational field of the point mass M $k = GM$). Using the representation (525), (526) the Hamiltonian can be written as

$$H_p = \frac{1}{2m}\left(p^2 - (\boldsymbol{\eta}\cdot\mathbf{L}) + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{4}\right) - \frac{mk}{\sqrt{x^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}} \tag{580}$$

Let us write an expansion for the Hamiltonian (580) over the parameters of noncommutativity. Note, that the operators under the square root

$$\sqrt{x^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{[\boldsymbol{\theta}\times\mathbf{p}]^2}{4}} \tag{581}$$

do not commute. Therefore, the expansion for X has an additional term $\theta^2 f(\mathbf{x})$ which is caused by the noncommutativity of x^2 and $[\boldsymbol{\theta}\times\mathbf{p}]^2$ [73]

$$X=\sqrt{x^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{[\boldsymbol{\theta}\times\mathbf{p}]^2}{4}}=x-\frac{1}{2x}(\boldsymbol{\theta}\cdot\mathbf{L})-\frac{1}{8x^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2+$$
$$\frac{1}{16}\left(\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2+[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\theta^2 f(\mathbf{x})\right) \tag{582}$$

where $f(\mathbf{x})$ is a function which can be found from

$$x^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2=x^2-(\boldsymbol{\theta}\cdot\mathbf{L})+$$
$$\frac{1}{16}\left(2[\boldsymbol{\theta}\times\mathbf{p}]^2+x[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2x+2x\theta^2 f(\mathbf{x})\right) \tag{583}$$

The equation (583) is obtained squaring the left- and right-hand sides of the equation (582). From (583), we have

$$\theta^2 f(\mathbf{x})=\frac{\hbar^2}{x^5}[\boldsymbol{\theta}\times\mathbf{x}]^2 \tag{584}$$

Therefore, the expansion for X reads [73]

$$X=x-\frac{1}{2x}(\boldsymbol{\theta}\cdot\mathbf{L})-\frac{1}{8x^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2+\frac{1}{16}\left(\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2+[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\frac{\hbar^2}{x^5}[\boldsymbol{\theta}\times\mathbf{x}]^2\right) \tag{585}$$

On the basis of (585) for $1/X$ up to the second order in the parameter $\boldsymbol{\theta}$ we have the following expansion

$$\frac{1}{X}=\frac{1}{\sqrt{x^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}=\frac{1}{x}+\frac{1}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{3}{8x^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2-$$
$$\frac{1}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x^2}+\frac{\hbar^2}{x^7}[\boldsymbol{\theta}\times\mathbf{x}]^2\right) \tag{586}$$

Thus, up to the second order in the parameters of noncommutativity the Hamiltonian H_p can be written as

$$H_p=\frac{p^2}{2m}-\frac{km}{x}-\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m}+\frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}-\frac{km}{2x^3}(\boldsymbol{\theta}\cdot\mathbf{L})-\frac{3km}{8x^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2+$$
$$\frac{km}{16}\left(\frac{1}{x^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x}+\frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x^2}+\frac{\hbar^2}{x^7}[\boldsymbol{\theta}\times\mathbf{x}]^2\right) \tag{587}$$

The total Hamiltonian reads $H = H_p + H^a_{osc} + H^b_{osc} = H_0 + \Delta H$ (see (510), (511), (512)) where

$$H_0 = \frac{p^2}{2m} - \frac{km}{x} + \frac{\langle\eta^2\rangle x^2}{12m} - \frac{kmL^2\langle\theta^2\rangle}{8x^5} + \frac{km\langle\theta^2\rangle}{24}\left(\frac{2}{x^3}p^2 + \frac{6i\hbar}{x^5}(\mathbf{x}\cdot\mathbf{p}) - \frac{\hbar^2}{x^5}\right) + H^a_{osc} + H^b_{osc} \tag{588}$$

$$\Delta H = -\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m} + \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m} - \frac{\langle\eta^2\rangle x^2}{12m} - \frac{km}{2x^3}(\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{kmL^2\langle\theta^2\rangle}{8x^5} + \frac{km}{16}\left(\frac{1}{x^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x} + \frac{1}{x}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{x^2} + \frac{\hbar^2}{x^7}[\boldsymbol{\theta}\times\mathbf{x}]^2\right) - \frac{3km}{8x^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 - \frac{km\langle\theta^2\rangle}{24}\left(\frac{1}{x^2}p^2\frac{1}{x} + \frac{1}{x}p^2\frac{1}{x^2} + \frac{\hbar^2}{x^5}\right) \tag{589}$$

Up to the second order in ΔH (or taking into account (589) up to the second order in the parameters of noncommutativity) the motion of a particle in a gravitational field is described by (588) and the equations of motion are the following [122]

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m} - \frac{km\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\mathbf{p} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p})\right) \tag{590}$$

$$\dot{\mathbf{p}} = -\frac{km\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m} - \frac{km\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p})\mathbf{p} - \frac{2\mathbf{x}}{x^5}p^2 + \frac{5\mathbf{x}}{2x^7}L^2 + \frac{5\hbar^2\mathbf{x}}{6x^7} - \frac{5i\hbar}{x^7}\mathbf{x}(\mathbf{x}\cdot\mathbf{p})\right) \tag{591}$$

In the classical limit $\hbar \to 0$ equations (590), (591) reduce to

$$\dot{\mathbf{x}} = \mathbf{p}' - \frac{km^2\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\mathbf{p}' - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p}')\right) \tag{592}$$

$$\dot{\mathbf{p}}' = -\frac{k\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m^2} - \frac{km^2\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p}')\mathbf{p}' - \frac{2\mathbf{x}}{x^5}(p')^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\mathbf{p}']^2\right) \tag{593}$$

where $\mathbf{p}' = \mathbf{p}/m$. The equations of motion of a particle in a gravitational field depend on the values $m^2\langle\theta^2\rangle$, $\langle\eta^2\rangle/m^2$. Therefore, the weak equivalence principle is violated. Considering the conditions (551), (552) we can write

$$\langle\theta^2\rangle m^2 = \frac{3\tilde{\gamma}^2}{2l_P^2} = A \tag{594}$$

(A is a constant which does not depend on mass) and also (572), therefore, the equations of motion read

$$\dot{\mathbf{x}} = \mathbf{p}' - \frac{kA}{12}\left(\frac{1}{x^3}\mathbf{p}' - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p}')\right) \tag{595}$$

$$\dot{\mathbf{p}}' = -\frac{k\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \frac{kA}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p}')\mathbf{p}' - \frac{2\mathbf{x}}{x^5}(p')^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\mathbf{p}']^2\right) \tag{596}$$

The constants A, B are the same for particles with different masses. Thus, analyzing the equations (595), (596) we can conclude that the weak equivalence principle is preserved in the noncommutative phase space.

Note also that if the relations (551), (552) are satisfied, the equations of motion in the quantum case (590), (591) read

$$\dot{\mathbf{x}} = \mathbf{p}' - \frac{kB}{12}\left(\frac{1}{x^3}\mathbf{p}' - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p}')\right) \tag{597}$$

$$\dot{\mathbf{p}}' = -\frac{k\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} -$$

$$\frac{kA}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p}')\mathbf{p}' - \frac{2\mathbf{x}}{x^5}(p')^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\mathbf{p}']^2 + \frac{5\hbar^2\mathbf{x}}{6m^2x^7} - \frac{5i\hbar}{mx^7}\mathbf{x}(\mathbf{x}\cdot\mathbf{p}')\right) \tag{598}$$

Due to the commutation relation

$$[x_i, p'_j] = i\delta_{ij}\frac{\hbar}{m} \tag{599}$$

these equations depend on $\hbar/m$, as it has to be [148].

Similarly, for a body of mass M in a gravitational field we can write

$$H_s = \frac{(P^c)^2}{2M} - \frac{kM}{(X^c)^2} + H_{rel} \tag{600}$$

$$H_0 = H_{cm} + \langle H_{rel}\rangle_{ab} + H^a_{osc} + H^b_{osc} \tag{601}$$

$$H_{cm} = \frac{(p^c)^2}{2M} - \frac{kM}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} - \frac{kM(L^c)^2\langle\theta^2\rangle}{8(x^c)^5} +$$

$$\frac{kM\langle(\theta^c)^2\rangle}{24}\left(\frac{2}{(x^c)^3}(p^c)^2 + \frac{6i\hbar}{(x^c)^5}(\mathbf{x}^c\cdot\mathbf{p}^c) - \frac{\hbar^2}{(x^c)^5}\right) \tag{602}$$

If the conditions (551), (552) are satisfied we have (577) and

$$\langle(\theta^c)^2\rangle = \frac{3\tilde{\gamma}^2}{2l_P^2M^2} = \frac{A}{M^2} \tag{603}$$

In this case the equations of motion of a macroscopic body in a non-uniform gravitational field read

$$\dot{\mathbf{x}}^c = \mathbf{p}^{c\prime} - \frac{kB}{12}\left(\frac{1}{(x^c)^3}\boldsymbol{p}^{c\prime} - \frac{3\mathbf{x}^c}{(x^c)^5}(\mathbf{x}^{\mathbf{c}}\cdot\mathbf{p}^{c\prime})\right) \tag{604}$$

$$\dot{\boldsymbol{v}}^c = -\frac{k\mathbf{x}^c}{(x^c)^3} - \frac{B\mathbf{x}^c}{6} -$$

$$\frac{kA}{4}\left(\frac{1}{(x^c)^5}(\mathbf{x}^c\cdot\mathbf{p}^{c\prime})\mathbf{p}^{c\prime}-\frac{2\mathbf{x}^c}{(x^c)^5}(p^{c\prime})^2+\frac{5\mathbf{x}^c}{2(x^c)^7}[\mathbf{x}^c\times\mathbf{p}^{c\prime}]^2\right) \tag{605}$$

Hence, the equations of motion do not depend on mass and the composition of the body and the weak equivalence principle is preserved [122, 119].

6. Many-particle problem in Lie-algebraic deformed space

The idea to relate the parameters of a deformed algebra to mass is also important for solving the problem of a macroscopic body and the problem of violation of the weak equivalence principle in spaces with the Lie-algebraic noncommutativity [82–86].

We analyze the Poisson brackets for the coordinates and momenta of the center-of-mass of a composite system in the frame of different noncommutative algebras of the Lie-type (space coordinates commute to time, space coordinates commute to space, a general case of the noncommutative algebra of the Lie type). These analyses are presented in Section 6.1. Section 6.2 is devoted to studies of the weak equivalence principle in the Lie-deformed space.

6.1. Composite system in space with Lie algebraic noncommutativity

Let us first consider a noncommutative algebra of the Lie type characterized by the relations (134)–(136) and study a general case when the coordinates and momenta of different particles $X_i^{(a)}$, $P_i^{(a)}$ satisfy the noncommutative algebra with different parameters κ_a (index a labels the particles). In the limit $\hbar\to 0$ we can write the following Poisson brackets [82]

$$\{X_i^{(a)},X_j^{(b)}\}=\frac{t}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau}-\delta_{i\tau}\delta_{j\rho}\right)\delta_{ab} \tag{606}$$

$$\{X_i^{(a)},P_j^{(b)}\}=\delta_{ab}\delta_{ij} \tag{607}$$

$$\{P_i^{(a)},P_j^{(b)}\}=0 \tag{608}$$

For the coordinates and momenta of the center-of-mass and the coordinates and momenta of the relative motion defined in the traditional way (259), (260), (348), (349), taking into account (606)–(608) we find

$$\{\tilde{X}_i,\tilde{X}_j\}=t\sum_a\frac{\mu_a^2}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau}-\delta_{i\tau}\delta_{j\rho}\right) \tag{609}$$

$$\{\tilde{X}_i,\tilde{P}_j\}=\delta_{ij},\quad \{\tilde{P}_i,\tilde{P}_j\}=0 \tag{610}$$

$$\{\Delta X_i^{(a)},\Delta X_j^{(b)}\}=t\left(\frac{\delta^{ab}}{\kappa_a}-\frac{\mu_a}{\kappa_a}-\frac{\mu_b}{\kappa_b}+\sum_c\frac{\mu_c^2}{\kappa_c}\right)\times\left(\delta_{i\rho}\delta_{j\tau}-\delta_{i\tau}\delta_{j\rho}\right) \tag{611}$$

$$\{\Delta X_i^{(a)},\Delta P_j^{(b)}\}=\delta_{ab}-\mu_b \tag{612}$$

$$\{\Delta X_i^{(a)}, \tilde{X}_j\} = t\left(\frac{\mu_a}{\kappa_a} - \sum_c \frac{\mu_c^2}{\kappa_c}\right)\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right) \tag{613}$$

$$\{\Delta P_i^{(a)}, \Delta P_i^{(b)}\} = \{\tilde{P}_i, \Delta P_j^{(b)}\} = 0 \tag{614}$$

Note that the coordinates of the center-of-mass satisfy the noncommutative algebra with an effective parameter of noncommutativity

$$\tilde{\theta}_{ij}^0 = \sum_a \frac{\mu_a^2}{\kappa_a}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{j\tau}\delta_{i\rho}\right) = \frac{1}{\kappa_{eff}}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{i\tau}\delta_{j\rho}\right) \tag{615}$$

where

$$\frac{1}{\kappa_{eff}} = \sum_a \frac{\mu_a^2}{\kappa_a} \tag{616}$$

Similarly as in the case of the canonical version of a noncommutative algebra there is a reduction in the effective parameter of noncommutativity which corresponds to a composite system with respect to the parameters of noncommutativity corresponding to individual particles. For a system of particles with the same masses and parameters of noncommutativity the effective parameter $\tilde{\theta}_{ij}^0$ decreases with the increasing number of particles N, namely $\tilde{\theta}_{ij}^0 = \left(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\right)/N\kappa$ [81].

Also it is important to mention that the motion of the center-of-mass is not independent of the relative motion because of the Poisson brackets (613) [81]. Assuming that the parameter of the noncommutative algebra κ_a depends on mass as

$$\frac{\kappa_a}{m_a} = \gamma_\kappa = const \tag{617}$$

(here γ_κ is a constant which is the same for different particles), we have $\{\Delta X_i^{(a)}, \tilde{X}_j\} = 0$, therefore, the relative motion has no influence on the motion of the center-of-mass. In addition, due to the relation (617) the effective parameter can be written as

$$\tilde{\theta}_{ij}^0 = \frac{1}{\gamma_\kappa M}\left(\delta_{i\rho}\delta_{j\tau} - \delta_{j\tau}\delta_{i\rho}\right) \tag{618}$$

Note that $\tilde{\theta}_{ij}^0$ does not depend on the composition of a system and is determined by its total mass M. On the basis of the expressions (615), (618) we obtain the relation (617) for the effective parameter of noncommutativity $\kappa_{eff} = \gamma_\kappa M$.

Let us consider another case of the noncommutative algebra of the Lie type. Namely, let us study the case when the space coordinates commute to space (137)–(140). For the coordinates and momenta of particles we have

$$\{X_k^{(a)}, X_\gamma^{(b)}\} = \delta_{ab}\frac{X_l^{(a)}}{\tilde{\kappa}}, \quad \{X_l^{(a)}, X_\gamma^{(b)}\} = -\delta_{ab}\frac{X_k^{(a)}}{\tilde{\kappa}} \tag{619}$$

$$\{P_k^{(a)}, X_\gamma^{(b)}\} = \delta_{ab}\frac{P_l^{(a)}}{\tilde{\kappa}}, \quad \{P_l^{(a)}, X_\gamma^{(b)}\} = -\delta_{ab}\frac{P_k^{(a)}}{\tilde{\kappa}} \tag{620}$$

$$\{X_i^{(a)}, P_j^{(b)}\} = \delta_{ab}\delta_{ij}, \quad \{X_\gamma^{(a)}, P_\gamma^{(b)}\} = \delta_{ab} \tag{621}$$

$$\{X_k^{(a)}, X_l^{(b)}\} = \{P_m^{(a)}, P_n^{(b)}\} = 0 \tag{622}$$

where $\tilde{\kappa}$ is a constant, the indexes a,b label the particles, k, l, γ are different and fixed, $k,l,\gamma = (1,2,3)$, $i \neq \gamma$, $j \neq \gamma$ and $m,n = (1,2,3)$, [82]. For the coordinates and momenta of the center-of-mass, the coordinates and momenta of the relative motion defined traditionally (259), (260), (348), (349), taking into account that $X_i^{(a)}$, $P_i^{(a)}$ satisfy the relations (619)–(622) we obtain

$$\{\tilde{X}_k, \tilde{X}_\gamma\} = \sum_a \frac{\mu_a^2 X_l^{(a)}}{\tilde{\kappa}_a}, \quad \{\tilde{X}_l, \tilde{X}_\gamma\} = -\sum_a \frac{\mu_a^2 X_k^{(a)}}{\tilde{\kappa}_a} \tag{623}$$

$$\{\tilde{P}_k, \tilde{X}_\gamma\} = \sum_a \frac{\mu_a P_l^{(a)}}{\tilde{\kappa}_a}, \quad \{\tilde{P}_l, \tilde{X}_\gamma\} = -\sum_a \frac{\mu_a P_k^{(a)}}{\tilde{\kappa}_a} \tag{624}$$

$$\{\tilde{X}_i, \tilde{P}_j\} = \delta_{ij}, \quad \{\tilde{X}_\gamma, \tilde{P}_\gamma\} = 1 \quad \{\tilde{X}_k, \tilde{X}_l\} = \{\tilde{P}_m, \tilde{P}_n\} = 0 \tag{625}$$

It is important to stress that the relations for the coordinates and momenta of the center-of-mass (623), (624) do not correspond to the relations of the noncommutative algebra (619), (620). There are no coordinates and momenta of the center-of-mass in the right-hand side of the relations (623), (624).

If we consider the parameter of the noncommutative algebra (619)–(622) to be dependent on mass as

$$\frac{\tilde{\kappa}_a}{m_a} = \gamma_{\tilde{\kappa}} = const \tag{626}$$

(here $\gamma_{\tilde{\kappa}}$ does not depend on mass) we can write

$$\{\tilde{X}_k, \tilde{X}_\gamma\} = \frac{1}{\tilde{\kappa}_{eff}}\tilde{X}_l, \quad \{\tilde{X}_l, \tilde{X}_\gamma\} = -\frac{1}{\tilde{\kappa}_{eff}}\tilde{X}_k, \quad \{\tilde{X}_k, \tilde{X}_l\} = 0 \tag{627}$$

$$\{\tilde{P}_k, \tilde{X}_\gamma\} = \frac{\tilde{P}_l}{\tilde{\kappa}_{eff}}, \quad \{\tilde{P}_l, \tilde{X}_\gamma\} = -\frac{\tilde{P}_k}{\tilde{\kappa}_{eff}} \tag{628}$$

where we use the notation $\tilde{\kappa}_{eff} = \gamma_{\tilde{\kappa}} M$. The relations (627), (628) reproduce the relations of the noncommutative algebra (619), (620) [81].

For the coordinates and momenta of the center-of-mass we can also calculate

$$\begin{gathered}
\{\Delta X_k^{(a)}, \tilde{X}_\gamma\} = \{\tilde{X}_k, \Delta X_\gamma^{(a)}\} = \frac{\mu_a X_l^{(a)}}{\tilde{\kappa}_a} - \sum_b \frac{\mu_b^2 X_l^{(b)}}{\tilde{\kappa}_b} \\
\{\Delta X_l^{(a)}, \tilde{X}_\gamma\} = \{\tilde{X}_l, \Delta X_\gamma^{(a)}\} = -\frac{\mu_a X_k^{(a)}}{\tilde{\kappa}_a} + \sum_b \frac{\mu_b^2 X_k^{(b)}}{\tilde{\kappa}_b} \\
\{\Delta X_k^{(a)}, \tilde{X}_l\} = \{\Delta X_l^{(a)}, \tilde{X}_k\} = 0
\end{gathered} \tag{629}$$

$$
\begin{aligned}
&\{\tilde{P}_k, \Delta X_\gamma^{(a)}\} = \frac{P_l^{(a)}}{\tilde{\kappa}_a} - \sum_b \frac{\mu_b P_l^{(b)}}{\tilde{\kappa}_b} \\
&\{\tilde{P}_l, \Delta X_\gamma^{(a)}\} = -\frac{P_k^{(a)}}{\tilde{\kappa}_a} + \sum_b \frac{\mu_b P_k^{(b)}}{\tilde{\kappa}_b} \\
&\{\Delta P_k^{(a)}, \tilde{X}_\gamma\} = \mu_a \left(\frac{P_l^{(a)}}{\tilde{\kappa}_a} - \sum_b \frac{\mu_b P_l^{(b)}}{\tilde{\kappa}_b} \right) \\
&\{\Delta P_l^{(a)}, \tilde{X}_\gamma\} = -\mu_a \left(\frac{P_k^{(a)}}{\tilde{\kappa}_a} - \sum_b \frac{\mu_b P_k^{(b)}}{\tilde{\kappa}_b} \right) \\
&\{\tilde{P}_k, \Delta X_l^{(a)}\} = \{\tilde{P}_l, \Delta X_k^{(a)}\} = \{\Delta P_l^{(a)}, \tilde{X}_k\} = \{\Delta P_k^{(a)}, \tilde{X}_l\} = 0
\end{aligned} \tag{630}
$$

Note that in the space with the Lie-algebraic noncommutativity (619)–(622) the motion of the center-of-mass depends on the relative motion, even if the condition (626) is satisfied. In this case we have

$$
\begin{aligned}
&\{\Delta X_k^{(a)}, \tilde{X}_\gamma\} = \{\tilde{X}_k, \Delta X_\gamma^{(a)}\} = \frac{1}{\tilde{\kappa}_{eff}} \Delta X_l^{(a)} \\
&\{\Delta X_l^{(a)}, \tilde{X}_\gamma\} = \{\tilde{X}_l, \Delta X_\gamma^{(a)}\} = -\frac{1}{\tilde{\kappa}_{eff}} \Delta X_k^{(a)} \\
&\{\tilde{P}_k, \Delta X_\gamma^{(a)}\} = \frac{1}{\tilde{\kappa}_a} \Delta P_l^{(a)}, \quad \{\tilde{P}_l, \Delta X_\gamma^{(a)}\} = -\frac{1}{\tilde{\kappa}_a} \Delta P_k^{(a)} \\
&\{\Delta P_k^{(a)}, \tilde{X}_\gamma\} = \frac{1}{\tilde{\kappa}_{eff}} \Delta P_l^{(a)}, \quad \{\Delta P_l^{(a)}, \tilde{X}_\gamma\} = -\frac{1}{\tilde{\kappa}_{eff}} \Delta P_k^{(a)}
\end{aligned} \tag{631}
$$

It is worth mentioning that the relation of the parameters of the noncommutative algebra with mass is also important in the frame of the generalized noncommutative algebra of the Lie type (141)–(142). Taking into account that in a general case the coordinates and momenta of different particles may satisfy the noncommutative algebra with different parameters we can write

$$
\{X_i^{(a)}, X_j^{(b)}\} = \delta_{ab} \theta_{ij}^{0(a)} t + \delta_{ab} \theta_{ij}^{k(a)} X_k^{(a)} \tag{632}
$$

$$
\{X_i^{(a)}, P_j^{(b)}\} = \delta_{ab} \delta_{ij} + \delta_{ab} \bar{\theta}_{ij}^{k(a)} X_k^{(a)} + \delta_{ab} \tilde{\theta}_{ij}^{k(a)} P_k^a
$$

$$
\{P_i^{(a)}, P_j^{(b)}\} = 0 \tag{633}
$$

In the noncommutative space characterized by the relations (632)–(633) the Poisson brackets for the coordinate and momenta of the center-of-mass (259), (260) read

$$
\{\tilde{X}_i, \tilde{X}_j\} = \sum_a \mu_a^2 \theta_{ij}^{0(a)} t + \sum_a \mu_a^2 \theta_{ij}^{k(a)} X_k^{(a)} \tag{634}
$$

$$\{\tilde{X}_i,\tilde{P}_j\}=\delta_{ij}+\sum_a \mu_a\bar{\theta}^{k(a)}_{ij}X^{(a)}_k+\sum_a \mu_a\tilde{\theta}^{k(a)}_{ij}P^a_k \tag{635}$$

$$\{\tilde{P}_i,\tilde{P}_j\}=0 \tag{636}$$

On the basis of the results of studies of a composite system in spaces (606)–(608), (619)–(622) we can conclude that the algebra for the coordinates and momenta of the center-of-mass reproduce the noncommutative algebra (632)–(633), if the parameters of noncommutativity satisfy the following relations

$$\theta^{0(a)}_{ij}m_a=\gamma^0_{ij}=const,\quad \theta^{k(a)}_{ij}m_a=\gamma^k_{ij}=const \tag{637}$$

$$\tilde{\theta}^{k(a)}_{ij}m_a=\tilde{\gamma}^k_{ij}=const \tag{638}$$

$$\bar{\theta}^{k(a)}_{ij}=\bar{\theta}^k_{ij} \tag{639}$$

with constants γ^0_{ij}, γ^k_{ij}, $\tilde{\gamma}^k_{ij}$ being antisymmetric to lower indexes and being the same for particles with different masses, and parameters $\bar{\theta}^k_{ij}$ are the same for different particles

Namely, if the relations (637)–(639) are satisfied we have

$$\{\tilde{X}_i,\tilde{X}_j\}=\theta^{0(eff)}_{ij}t+\theta^{k(eff)}_{ij}\tilde{X}_k \tag{640}$$

$$\{\tilde{X}_i,\tilde{P}_j\}=\delta_{ij}+\bar{\theta}^k_{ij}\tilde{X}_k+\tilde{\theta}^{k(eff)}_{ij}\tilde{P}_k \tag{641}$$

where

$$\theta^{0(eff)}_{ij}=\frac{\gamma^0_{ij}}{M},\quad \theta^{k(eff)}_{ij}=\frac{\gamma^k_{ij}}{M},\quad \tilde{\theta}^{k(eff)}_{ij}=\frac{\tilde{\gamma}^k_{ij}}{M} \tag{642}$$

and $M=\sum_a m_a$ [81].

In the particular cases of the noncommutative Lie algebra (143)–(146), (148)–(153) the conditions (637), (638) can be rewritten as (617), (626) and it follows from (639) that

$$\bar{\kappa}_a=\bar{\kappa} \tag{643}$$

In the next section we study the motion of a particle (body) in a gravitational field and we show that the relation of parameters of the noncommutative algebras with mass is also important for recovering the weak equivalence principle in a space with the Lie-algebraic noncommutativity.

6.2. Weak equivalence principle in the frame of noncommutative algebra of Lie type

In general, the noncommutativity of the Lie-type causes violation of the weak equivalence principle. Let us first examine the weak equivalence principle in the space characterized by (134)–(136).

In a space with coordinates commuting to time (134)–(136) for a particle with mass m in the gravitational field $V(X_1,X_2,X_3)$

$$H=\frac{\mathbf{P}^2}{2m}+mV(X_1,X_2,X_3) \tag{644}$$

taking into account (134)–(136), we have the following equations of motion

$$\dot{X}_i=\{X_i,H\}=\frac{P_i}{m}+\frac{tm}{\kappa}\frac{\partial V}{\partial X_k}\left(\delta_{i\rho}\delta_{k\tau}-\delta_{i\tau}\delta_{k\rho}\right) \tag{645}$$

$$\dot{P}_i=\{P_i,H\}=-m\frac{\partial V}{\partial X_i} \tag{646}$$

Note that even if the inertial mass is equal to the gravitational mass (see Hamiltonian (644)) because of the noncommutativity the motion of a particle in a gravitational field depends on its mass.

The situation is changed, if the condition (617) is satisfied. In this case we can write

$$\dot{X}_i=P'_i+\frac{t}{\gamma_\kappa}\frac{\partial V}{\partial X_k}\left(\delta_{i\rho}\delta_{k\tau}-\delta_{i\tau}\delta_{k\rho}\right) \tag{647}$$

$$\dot{P}'_i=-\frac{\partial V}{\partial X_i} \tag{648}$$

where $P'_i=P_i/m$. It follows from (647), (648) that the weak equivalence principle is satisfied [81].

Let us consider another case of a noncommutative algebra of the Lie type (619)–(622). In the case when the space coordinates commute to space (619)–(622) for a particle in the gravitational field (644) we have the following equations

$$\dot{X}_k=\frac{P_k}{m}+\frac{mX_l}{\tilde{\kappa}}\frac{\partial V}{\partial X_\gamma},\quad \dot{X}_l=\frac{P_l}{m}-\frac{mX_k}{\tilde{\kappa}}\frac{\partial V}{\partial X_\gamma} \tag{649}$$

$$\dot{X}_\gamma=\frac{P_\gamma}{m}-\frac{mX_l}{\tilde{\kappa}}\frac{\partial V}{\partial X_k}+\frac{mX_k}{\tilde{\kappa}}\frac{\partial V}{\partial X_l} \tag{650}$$

$$\dot{P}_k=-m\frac{\partial V}{\partial X_k}+\frac{mP_l}{\tilde{\kappa}}\frac{\partial V}{\partial X_\gamma},\quad \dot{P}_l=-m\frac{\partial V}{\partial X_l}-\frac{mP_k}{\tilde{\kappa}}\frac{\partial V}{\partial X_\gamma} \tag{651}$$

$$\dot{P}_\gamma=-m\frac{\partial V}{\partial X_\gamma} \tag{652}$$

Analyzing (649)–(652) we can conclude that the weak equivalence principle is violated because of the relations (649)–(652). If we consider the condition (626), we can rewrite (649)–(652) as

$$\dot{X}_k=P'_k+\frac{X_l}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_\gamma},\quad \dot{X}_l=P'_l-\frac{X_k}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_\gamma} \tag{653}$$

$$\dot{X}_\gamma = P'_\gamma - \frac{X_l}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_k} + \frac{X_k}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_l} \tag{654}$$

$$\dot{P}'_k = -\frac{\partial V}{\partial X_k} + \frac{P'_l}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_\gamma}, \quad \dot{P}'_l = -\frac{\partial V}{\partial X_l} - \frac{P'_k}{\gamma_{\tilde{\kappa}}}\frac{\partial V}{\partial X_\gamma} \tag{655}$$

$$\dot{P}'_\gamma = -\frac{\partial V}{\partial X_\gamma} \tag{656}$$

$(P'_i = P_i/m)$. Hence, the weak equivalence principle is preserved [81].

In the general case of the noncommutative algebra of the Lie type (141)–(142) the equations of motion for a particle in the gravitational field (355) read

$$\dot{X}_i = \frac{P_i}{m} + \bar{\theta}^k_{ij}\frac{P_j X_k}{m} + \tilde{\theta}^k_{ij}\frac{P_j P_k}{m} + m(\theta^0_{ij}t + \theta^k_{ij}X_k)\frac{\partial V}{\partial X_j} \tag{657}$$

$$\dot{P}_i = -m\frac{\partial V}{\partial X_i} - m(\bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k)\frac{\partial V}{\partial X_j} \tag{658}$$

If the parameters θ^0_{ij}, θ^k_{ij}, $\tilde{\theta}^k_{ij}$ are related to mass as (637), (638) and the parameters $\bar{\theta}^k_{ij}$ satisfy the relation (639) using the notation $P'_i = P_i/m$ we can write

$$\dot{X}_i = P'_i + \bar{\theta}^k_{ij}P'_j X_k + \tilde{\gamma}^k_{ij}P'_j P'_k + (\gamma^0_{ij}t + \gamma^k_{ij}X_k)\frac{\partial V}{\partial X_j} \tag{659}$$

$$\dot{P}'_i = -\frac{\partial V}{\partial X_i} - (\bar{\theta}^k_{ij}X_k + \tilde{\gamma}^k_{ij}P'_k)\frac{\partial V}{\partial X_j} \tag{660}$$

Thus, in a general case of a noncommutative space (141)–(142) the problem of violation of the weak equivalence principle can be solved due to the conditions (637), (638), (639) [81].

The conclusions can be generalized to the case of motion of a macroscopic body in a gravitational field in a space with noncommutativity of the Lie type. In a general case of the noncommutative algebra of the Lie-type (141)–(142), for a body of mass M, considering the case when the conditions (637), (638), (639) are satisfied and the influence of the relative motion on the motion of the center-of-mass can be neglected, we can write the following equations of motion

$$\dot{\tilde{X}}_i = \tilde{P}'_i + \left(\bar{\theta}^k_{ij}\tilde{X}_k + \tilde{\gamma}^k_{ij}\tilde{P}'_k\right)\tilde{P}'_j + \left(\gamma^0_{ij}t + \gamma^k_{ij}\tilde{X}_k\right)\frac{\partial V}{\partial \tilde{X}_j} \tag{661}$$

$$\dot{\tilde{P}}'_i = -\frac{\partial V}{\partial \tilde{X}_i} - \left(\bar{\theta}^k_{ij}\tilde{X}_k + \tilde{\gamma}^k_{ij}\tilde{P}'_k\right)\frac{\partial V}{\partial \tilde{X}_j} \tag{662}$$

Thus, in the noncommutative space described by the relations (141)–(142) the weak equivalence principle is recovered, if the parameters of the algebra satisfy the relations (637), (638), (639) [81].

At the end of this chapter it is worth noting that due to the relation of parameters of the noncommutative algebra with mass, similar results can be obtained in a space with quadratic noncommutativity

$$\{X_k, X_\gamma\} = \frac{1}{\bar{\kappa}} t X_l, \quad \{X_l, X_\gamma\} = -\frac{1}{\bar{\kappa}} t X_k \tag{663}$$

$$\{X_k, X_l\} = 0, \quad \{P_n, P_m\} = 0 \tag{664}$$

$$\{X_\gamma, P_k\} = -\frac{1}{\bar{\kappa}} t P_l, \quad \{X_\gamma, P_l\} = \frac{1}{\bar{\kappa}} t P_k \tag{665}$$

$$\{X_i, P_j\} = \delta_{ij}, \quad \{X_\gamma P_\gamma\} = 1, \quad \{P_n, P_m\} = 0 \tag{666}$$

where indexes k, l,γ are fixed, $k \neq l \neq \gamma$, also $i \neq \gamma$, $j \neq \gamma$ and $n, m = (1,2,3)$ [82, 149], and in the twist deformed space characterized by

$$\{X_i, X_j\} = f\left(\frac{t}{\tau}\right)\theta_{ij} \tag{667}$$

$$\{X_i, P_j\} = i\hbar\delta_{ij} \tag{668}$$

$$\{P_i, P_j\} = 0 \tag{669}$$

where τ is a time-scale parameter, f is a function of time, the parameters θ_{ij} are considered to be constants, τ is a time scale parameter [150–153].

The soccer-ball problem is solved and the weak equivalence principle is preserved in the frame of the algebra (663)–(666), if the parameter $\bar{\kappa}_a$ is related to mass as

$$\frac{\bar{\kappa}_a}{m_a} = \gamma_{\bar{\kappa}} = const \tag{670}$$

where $\gamma_{\bar{\kappa}}$ is the same for particles with different masses [154]. In the frame of the twist deformed space (134)–(136) the weak equivalence principle is recovered, the motion of the center-of-mass of a composite system is independent of the relative motion, the coordinates can be considered as kinematic variables, if the parameters θ_{ij} are inversely proportional to mass (448) [155].

7. Conclusions

The problem of a macroscopic body known as the soccer-ball problem appears in the frame of different algebras, if we consider the relations of a deformed algebra to be the same for the coordinates and momenta of elementary particles and for the coordinates and momenta of macroscopic bodies. Namely, we face a problem of great influence of space quantization on the motion of macroscopic bodies, the problem of nonadditivity of the kinetic energy, its dependence on the composition, the problem of great violation of the weak equivalence principle, the problem of the dependence of the Galilean and Lorentz transformations on mass, the problem of extremely small results for the minimal length obtained on the basis of studies of the Mercury's perihelion shift [29, 97].

The commutation relations for the coordinates and momenta of the center-of-mass of a composite system (macroscopic body) in quantum space do not correspond to the commutation relations for coordinates and momenta of elementary particles. In the case of a two-dimensional noncommutative algebra of a canonical type (338)–(340) one obtains that the coordinates and momenta of the center-of-mass of a composite system satisfy the relations of a noncommutative algebra with the effective parameters of noncommutativity which depend on the masses of the particles forming the system. In a deformed space with minimal length (10)–(12), in a six-dimensional noncommutative phase space of a canonical type (49)–(51), in a space with the Lie-algebraic noncommutativity (133) the relations for the coordinates and momenta of the center-of-mass of a composite system do not reproduce the relations of the corresponding algebras.

It is important to mention that if the parameters of deformed algebras are considered to be different for different particles and to be dependent on their masses, the coordinates and momenta of the center-of-mass of a system satisfy the relations of deformed algebras with effective parameters of deformation which are determined by the total mass of the system. Besides, due to the idea to relate the parameters of the corresponding algebras to mass, the list of important results in the frame of different algebras (deformed algebra with minimal length, noncommutative algebras of a canonical type, algebras with noncommutativity of the Lie type, algebra with quadratic noncommutativity, a twist-deformed algebra) can be obtained. The problem of the great influence of space quantization on the macroscopic bodies does not appear among them, the properties of the kinetic energy are preserved, the weak equivalence principle is recovered. In addition, we show that due to the relation of parameters of deformation to mass in the deformed space with minimal length (3) the Galilean and Lorentz transformations are the same for particles with different masses.

Hence, the number of the results and the number of the algebras justify the importance of the idea to relate the parameters of deformed algebras to mass.

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Summary

The monograph is devoted to studies of the problem of a macroscopic body known as the soccer-ball problem in the frame of different deformed algebras leading to space quantization. It is shown that this problem can be solved in a deformed space with a minimal length, in a noncommutative phase space, in a space with a Lie-algebraic noncommutativity, in a twist-deformed space-time due to the relation of parameters of corresponding algebras with mass. In addition, we conclude that this relation gives a possibility to obtain a list of important results in quantum space including recovering the weak equivalence principle, preserving the properties of the kinetic energy, obtaining the Galilean and Lorentz transformations independent of the mass of the particle.

Keywords: quantum space, macroscopic body, weak equivalence principle, properties of kinetic energy, minimal length.

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